PRENILPOTENT SPACES.

By E. DROR and W. G. DWYER.

Introduction. Nilpotent spaces, as introduced in [3], have proven to be as easy to handle as simply connected spaces. Being much more general than the latter, they seem to provide the correct conceptual framework which was earlier assigned to simply connected spaces. Adams, Kan, Curtis and Bousfield, along with many other people, have investigated the relation between the homology groups, and operation thereon, and the homotopy type. For nilpotent spaces this relation is expressed by the existence of a stable and unstable Adams spectral sequence [2]. However, for general nonnilpotent spaces very little is known about the relation between homotopy and homology. The problems here are not only computational, since there are many different homotopy types which have the same homology groups and operations thereon.

We have indicated a general direction of analysis in the work on acyclic spaces and homology spheres [4,5]. In the course of the attempt to generalize that work to more complicated spaces, like knot complements, it became clear that one has to solve the following general problem.

Characterize all spaces X for which there is a map $f: X \to N$ such that:

(i) the induced map $H_*(f,Z): H_*(X,Z) \to H_*(N,Z)$ is an isomorphism;
(ii) the space N is nilpotent.

Such spaces will be called *prenilpotent.*

It is the aim of the present paper to solve this problem for spaces with finite number of cells in each dimension. Some applications will be given in a forthcoming paper on homology circles and knot complements. [10]

Organization of the Paper. Some basic, mostly algebraic, definitions and results are given in the first section. These will allow us to state there our main result. In Section 2 we give some examples and indicate some applications for the main result. The proof is given in Section 3. By H_k we always mean $H_k(\ ,Z)$.

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1. The Main Theorem.

1.1. *Preliminaries.* Let π be a group—in practice the fundamental group—and let G be a π-group, i.e., a group with π-action $\psi:\pi\to\operatorname{aut}G$. If $G\subseteq\pi$ we always take $\psi(x)g=x\cdot g\cdot x^{-1}$. By $\Gamma_2 G$ we mean (compare [3]) the normal π-subgroup of G generated by all elements of the form $(\psi(x)g)\cdot g^{-1}$, where $x\in\pi$ and $g\in G$. Inductively we define $\Gamma_n G=\Gamma_2(\Gamma_{n-1}G)$. A π-group G is called *nilpotent* if for some r one has $\Gamma_r G=0$; it is called (π)-perfect if $\Gamma_2 G=G$. A π-subgroup generated by a collection of π-perfect subgroups is itself π-perfect. Therefore one can define for every G a functorial subgroup ΓG to be the π-subgroup generated by all π-perfect subgroup of G. Thus ΓG is *the maximal* (largest) *π-perfect subgroup of G.*

In what follows X denotes a connected locally finite space, i.e., one with finitely many cells in each dimension. The fundamental group $\pi=\pi_1 X$ acts naturally on $\pi_* X$ and on the homology of any covering space $\tilde{X}$ of X. Thus $\Gamma\pi_1 X$ and $\Gamma H_n(\tilde{X})$ are functors on spaces which yield certain perfect π-groups. Let $\tilde{X}_\Gamma$ denote the cover of X which corresponds to the subgroup $\Gamma\pi_1 X$. Our main result reads:

THEOREM 1. *A connected locally finite space is prenilpotent if and only if*

(1) $\pi_1 X/\Gamma\pi_1 X$ *is nilpotent, and*
(2) $H_i\tilde{X}_\Gamma/\Gamma H_i\tilde{X}_\Gamma$ *is nilpotent for all* $i>0$.

In the proof of Theorem 1 we will need two algebraic results.

PROPOSITION 1. *Let R be a noetherian ring and ν a finitely generated nilpotent group. Then the group ring $R(\nu)$ is also noetherian.*

PROPOSITION 2. *Let ν be as in Proposition 1 and M a finitely generated module over the integral group ring $Z(\nu)$. If M is ν-perfect then $H_*(\nu,M)\approx 0$.*

Proofs. For Proposition 1 see [8, 9]. Proposition 2 is the main result of [7].

2. Remarks and Examples.

Definition 1. *It is reasonable to call a π-group G for which $G/\Gamma G$ is π-nilpotent a prenilpotent π-group. For such a group the nilpotent completion $\lim_{\substack{\leftarrow\\ s}} G/\Gamma_s G$ is a nilpotent group, as we shall see presently.*

Definition 2. *The topological analog to a prenilpotent group is a space whose nilpotent completion* [2] *is nilpotent. Or, equivalently, as we shall presently see, a space for which there is a map $X\to N$ into a nilpotent space N, which induces isomorphism on homology. Thus we call such a space a* prenilpotent *space.*

With Definitions 1 and 2 our main Theorem 1 takes the form:

THEOREM 2. *A locally finite space X is prenilpotent if and only if both $\pi_1 X$ and $H_*(\tilde{X}_\Gamma)$ are prenilpotent.*

2.1. Theorem 2 is the analog of the following theorem [2]:

A space X is nilpotent if and only if both $\pi_1 X$ and $H_*(\tilde{X})$ are nilpotent. Here $\tilde{X}$ denotes the universal cover of X.

2.2. Let us clarify the notions of nilpotent, perfect and prenilpotent with some examples. First, note that if P is a perfect π-group and N is π-nilpotent then any π-map $P \rightarrow N$ is the zero map. This is so because this map factors through $P/\Gamma_i P \rightarrow N/\Gamma_i N$ where the target is isomorphic to N for big enough i and the range is always zero.

PROPOSITION. *A π-group G is prenilpotent if and only if for some $r>0$ $\Gamma G = \Gamma_r G$ or, equivalently, $\Gamma_{r+1} G = \Gamma_r G$.*

Proof. Since $G/\Gamma_r G$ is nilpotent one side is clear. If, on the other hand, G is prenilpotent, i.e., $G/\Gamma G$ is nilpotent, then $G/\Gamma G$ is a quotient of $G/\Gamma_s G$ for s big enough. Thus $\Gamma_s G \subseteq \Gamma G$ and, therefore, $\Gamma_s G = \Gamma G$. In that case $\Gamma_s G$ is perfect and is equal to $\Gamma_{s+1} G$. If they are equal then again $\Gamma_s G \subseteq \Gamma G$ and, therefore, $\Gamma_s G = \Gamma G$-since $\Gamma G \subseteq \Gamma_s G$ for all s.

Example 1. Every finite G is prenilpotent. This is immediate from 2.2 since if G is finite it has a finite number of subgroups and therefore $\Gamma_{l+1} G = \Gamma_l G$ for some l.

2.3. *The Nilpotent Quotients Lemma.* Let us restate this lemma which is at the root of the relation between homology and nilpotency (see [3]).

Let $f: A \rightarrow A'$ be a map of π-modules. If $H_0(\pi, f)$ is bijective and $H_1(\pi, f)$ is surjective then $f/\Gamma_r: A/\Gamma_r A \rightarrow A'/\Gamma_r A'$ is an isomorphism.

2.4. *The Bousfield–Kan Spectral Sequence.* The Bousfield–Kan completion tower for X, $(R_n X)$ arose out of their attempt to understand the general *unstable* Adams spectral sequence, which for nilpotent spaces goes from $H_*(X, R)$ to the R-homotopy of X (see [2]). It is natural to ask under what condition does the spectral sequence, which is always defined, converge strongly and to what. Theorem 1 provides the following answer.

THEOREM. *Let X be a locally finite space and let $E^r_{p,q}(X)$ be the integral Bousfield–Kan spectral sequence* [2]. *Then the following statements are equivalent.*

(1) *$E^r_{p,q}(X)$ converges strongly to the homotopy groups of* $Z_\infty X = \lim_{\overleftarrow{s}} Z_s X$.

(2) *$Z_\infty X$ is nilpotent and $X \to Z_\infty X$ homology isomorphism.*

(3) *Conditions* (1) *and* (2) *of Theorem* 1.

Proof. The equivalence of (1) and (2) is proven in [2]. Assume (2), then X is by definition prenilpotent. Thus by Theorem 1, statement (3) holds. Assume now (3); i.e., X is prenilpotent.

Let $X \to N$ be homology isomorphism to a nilpotent space N. Then $Z_\infty X \to Z_\infty N$ is an equivalence. But $N \sim Z_\infty N$; in other words, (2) holds.

3. Proof of the Main Theorem.

Pro-objects.

In order to proceed with the proof let us recall certain ideas concerning the Bousfield–Kan, Z-completion tower. In [2] Bousfield-Kan associated to every space a functorial tower

$$(Z_s X)_{s \leqslant \infty} \cdots \to Z_s X \to Z_{s-1} \to \cdots \to Z_0 X = ZX$$

where ZX is the free abelian space generated by X. For every s the space $Z_s X$ is a nilpotent space. The tower as a whole is a sort of "best nilpotent approximation" of X. Formally, one may say that $(Z_s X)_s$ is the pronilpotents completion of X in the sense of Artin–Mazur [1].

We will regard the tower $(Z_s X)_s$ as a prospace, i.e., an object in the category defined by Artin–Mazur. The reader is referred to [6] as well as to [2], [7] for the basic facts about that category. In what follows, we will try to explicitly state every needed property of that category. The reason we have to get into that category of pro-object is that the principle property of the tower $(Z_s X)_s$ is most conveniently formulated in terms of proisomorphism. That property is Corollary 3.1 of [6]. For the ring of integers it reads as follows.

THEOREM 3. *For every space X and every $k \geqslant 0$, the natural map $i_k : H_k X \to \{H_k Z_s X\}_{s \geqslant 0}$ is a proisomorphism of progroups.*

Proof. See [6].

Remark. To say that i_k is a proisomorphism is to say, in that case, that for every k there exists an integer N such that for every $s > N$ the group $H_k Z_s X$ is (naturally) isomorphic to $H_k X \oplus J_s$ and $(J_s)_s$ is a trivial (zero) progroup. A progroup $\{G_s\}$ is trivial iff for every s there exist $s' \geqslant s$ so that the map $G_{s'} \to G_s$ is the zero map.

Thus Theorem 3 guarantees that every space whatsoever can be mapped by a homology isomorphism to a *tower* of nilpotent space. Our proof will show that under the condition of Theorem 1 that tower can be replaced by a single nilpotent space which is "equivalent" to the whole tower.

We can conclude this introduction to the proof by two useful lemmas about pro-objects.

LEMMA 1. *Let $(X_s)_s$ be a tower of spaces. Assume that for each n, the tower of groups $(\pi_n X_s)_s$ (n-fixed) is proisomorphic to some fixed group π_n. Then the tower $\{X_s\}_s$ is weakly equivalent to some fixed space X; i.e., for every k the Postnikov pieces $\{P_k X_s\}_s$ and $P_k X$ are proequivalent.*

Proof. Lemma 1 is proved very simply by the techniques of [1].

LEMMA 2. *$\{\cdots \to N_s \to N_{s-1} \to \cdots N_0\}$ is a tower of nilpotent groups (or π-groups). If $(N_s)_s$ is perfect in the tower sense, i.e., if the tower $(N_s/\Gamma_2 N_s)_s$ is protrivial, then $(N_s)_s$ itself is proisomorphic to zero.*

Proof. Since $(N_s/\Gamma_2 N_s)_s \approx 0$ one gets that $(\Gamma_2 N_s)_s \to (N_s)_s$ is a proisomorphism. Further it follows from the 5-lemma for progroups [2] that for every $l \geq 2$ $(N_s/\Gamma_l N_s)_s$ is a protrivial tower. Now given an s we show that there exists an $s' > s$ such that the map $N_{s'} \to N_s$ is the zero map. Let k be an integer with $\Gamma_k N_s = 0$. Since the tower $(N_s/\Gamma_k N_s)_s$ is protrivial there exists an s' so that $N_{s'}/\Gamma_k N_{s'} \to N_s/\Gamma_k N_s$ is zero. It follows that the map $N_{s'} \to N_s/\Gamma_k N_s = N_s$ is also zero.

3.1. *Proof of the Necessity in Theorem* 1. Assume that we have a map b: $X \to N$ from X to a nilpotent space N such that $H_* b$ is an isomorphism. We use the diagram of fibrations

$$\begin{array}{ccc} \tilde{X}_\Gamma & \xrightarrow{\tilde{b}} & \tilde{N} \\ \downarrow & & \downarrow \\ X & \longrightarrow & N \\ \downarrow p & & \downarrow \tilde{p} \\ K(\nu, 1) & \longrightarrow & K(\nu, 1) \end{array}$$

in which $\nu = \pi_1 X/\Gamma \pi_1 X$, observing that ν must be nilpotent since $\pi_1 X/\Gamma_r \pi_1 X \approx \pi_1 X/\Gamma_r \pi_1 N$ by Stallings theorem (see [3]), and $\Gamma_I \pi_1 N = 0$ for some I gives $\Gamma \pi_1 X = \Gamma_I \pi_1 X$ and $\nu = \pi_1/\Gamma_I X$ $(= \pi_1 X/\Gamma_I \pi_1 X)$. Hence, in fact, $\pi_1 N = \nu$. Now we must establish some finiteness condition which will allow us to use Proposition 2.

3.2. *The group* $H_n\tilde{X}_\Gamma$ *are all finitely generated as modules over the integral group ring* $Z(\nu)$.

The chain complex $C_n(\tilde{X}_\Gamma, Z)$ is certainly finitely generated as module over $Z(\nu)$ since X has only finitely many cells in each dimension. The differential $d: C_n(\tilde{X}_\Gamma) \to C_{n-1}(\tilde{X}_\Gamma)$ being a $Z(\nu)$-map, $H_n(C_*) = H_n\tilde{X}_\Gamma$ appears as a subquotient of a finitely generated module over the ring $Z(\nu)$, which is, by Proposition 1, noetherian. Therefore H_nC_* is finitely generated over $Z(\nu)$. We must show that $H_n\tilde{X}_\Gamma$ is prenilpotent, or equivalently, that $H_n\tilde{X}_\Gamma/\Gamma H_n\tilde{X}$ is nilpotent. We will show that, in fact, one has an isomorphism of ν-modules $H_j\tilde{X}_\Gamma/\Gamma H_j\tilde{X}_\Gamma \to H_j\tilde{N}$. This is proven by induction. Being true for $j=0$, assume the above isomorphism for $j<n$. Taking the long exact sequence of coefficients

$$\cdots \xrightarrow{\partial} H_r(\nu, \Gamma H_j\tilde{X}_\Gamma) \to H_r(\nu, H_j\tilde{X}_\Gamma) \to H_r(\nu, H_j\tilde{X}_\Gamma/\Gamma H_j\tilde{X}_\Gamma) \xrightarrow{\partial}$$

one gets that the maps on the right are all isomorphisms since, from claim 3.2 above, $\Gamma H_j\tilde{X}_\Gamma$ is finitely generated, and therefore, by Proposition 2, the groups on the left are all isomorphic to zero. This means that one has $H_r(\nu, H_j\tilde{X}_\Gamma) \approx H_r(\nu, H_j\tilde{N})$ for $0 \leqslant r < \infty$ and $0 \leqslant j < n$. Now one uses the comparison theorem for the Serre spectral sequence of the fibration diagram above. Since H_*b is an isomorphism we get

$$H_0(\nu, H_n\tilde{X}_\Gamma) \approx H_0(\nu, H_n\tilde{N})$$
$$H_1(\nu, H_n\tilde{X}_\Gamma) \to H_1(\nu, H_n\tilde{N}) \quad \text{is onto.}$$

Using the nilpotent quotients lemma 2.3 we get that for every r the isomorphism

$$H_n\tilde{X}_\Gamma/\Gamma_r H_n\tilde{X}_\Gamma \approx H_n\tilde{N}/\Gamma_r H_n\tilde{N}.$$

Now, $H_r\tilde{N}$ is a nilpotent ν-module since $p: N \to K(\nu, 1)$ is a nilpotent fibration. Thus for r big enough the quotient module on the right is isomorphic to $H_n\tilde{N}$, and thus for such r the submodule $\Gamma_r H_n\tilde{X}_\Gamma$ is perfect. Hence $H_n\tilde{X}$ is prenilpotent and the induction assumption holds for n.

3.3. *Proof Sufficiency in Theorem* 1. Our proof will relate to the following diagram:

$$\begin{array}{cccc} F = & \tilde{X}_\Gamma & \xrightarrow{\tilde{b}} & (F_s)_s \\ & \downarrow & & \downarrow \\ & X & \xrightarrow{b} & (Z_s X)_s \\ & p\downarrow & & p_s\downarrow \\ & K(\nu,1) & \longrightarrow & K(\nu,1) \end{array}$$

where ν is the nilpotent group $\pi_1 X/\Gamma\pi_1 X$, $F = \tilde{X}_\Gamma$ is the fibre of the canonical map p. The map $(p_s)_s$ comes from the map $Z_s(p):(Z_sX)_s \to Z_sK(\nu,1)$ by replacing $Z_sK(\nu_1,1)$ by $K(\nu,1)$ since it is a nilpotent space with one nonvanishing homotopy group, and thus $K(\nu,1) \to Z_sK(\nu,1)$ is a proequivalence and has a homotopy inverse. For every s one takes F_s to be the (homotopy) fibre of p_s. Now since π_1 is prenilpotent, every p_s induces surjective map on π_1 and thus for all s, the fibre F_s is connected. Therefore, p_s for all s is a *nilpotent fibration*; i.e., ν acts nilpotentely on H_iF_s for all $i \geqslant 0$ [2]. Thus the map $H_*\tilde{b}$ sends ΓH_*F to zero by Remark 2.2 (ii), and $\tilde{b}$ induces a well defined map

$$\begin{array}{ccc} & & \vdots \\ & \nearrow & \downarrow \\ & & H_nF_s \\ & \nearrow & \downarrow \\ H_n\tilde{b}: H_nF/\Gamma H_nF & \longrightarrow & H_nF_{s-1} \\ & \searrow & \downarrow \\ & & \vdots \end{array}$$

of $H_nF/\Gamma H_nF$ to the tower $(H_nF_s)_s$. The central step of the proof is to show the *claim*: *The map $H_*\tilde{b}$ is a proisomorphism.*

The claim means, in particular, that $(H_nF_s)_s$ is proisomorphic to a fixed group which will imply that $(\pi_nF_s)_s$ is, too.

Proof of Claim. The proof goes by induction on n, and the rest on the comparison theorem for first quadrant spectral sequences with proisomorphic E^∞_{pq}-term. See [2].

The claim certainly holds for $n=0$. Assume that $H_j\tilde{b}$ is a proisomorphism for all $j<n$. One has the exact sequence of ν-modules $0 \to \Gamma H_jF \to H_jF \to H_jF/$

$\Gamma H_jF \to 0$ which gives a long exact sequence.

$$\cdots \to H_l(\nu, \Gamma H_jF) \to H_l(\nu, H_jF) \to H_l(\nu, H_jF/\Gamma H_jF) \to H_{l-1}(\nu, \Gamma H_jF) \to \cdots.$$

We have shown above that the modules H_jF are finitely generated over the group ring $Z(\nu)$. Since this group ring is noetherian, the submodules ΓH_jF are also finitely generated and thus one can apply Lemma 1.2 and get that the term $H_l(\nu, \Gamma H_jF) \approx 0$ for all l and all j. Combining the long exact sequence with the induction assumption one gets the proisomorphism

$$H_l(\nu, H_jF) \to (H_l(\nu, H_jF_s))_s$$

for all $l \geqslant 0$ and $0 \leqslant j < n$. It follows then from the comparison theorem for the Senne prospectral sequences of the fibrations p and $(p_s)_s$ that

$$H_0(\nu, H_nF) \to (H_0(\nu, H_nF_s))_s \quad \text{is bijective and}$$
$$H_1(\nu, H_nF) \to (H_1(\nu, H_nF_s))_s \quad \text{is onto.}$$

Using lemma 2.3 for the progroup case one gets that for every $r \geqslant 2$ one has a proisomorphism

$$H_nF/\Gamma_rH_nF \to (H_nF_s/\Gamma_rH_nF_s)_s.$$

By assumption, for I big enough $\Gamma_IH_nF = \Gamma H_nF$. Therefore $(\Gamma_{I+1}H_nF_s)_s \approx (\Gamma_IH_nF_s)_s$ is a perfect as a progroup over ν. But we saw that p_s is nilpotent and thus $\Gamma_IH_nF_s$ is nilpotent for every s, and hence by Lemma 2 the tower $(\Gamma_IH_nF_s)_s$ is protrivial. Taking $r = I$ one gets that $\tilde{b}_n$ is a proisomorphism. This completes the proof of the claim.

Next we note that by Lemma 2 $(\pi_1F_s)_s$ is protrivial.

It is a tower of nilpotent groups which is perfect in the prosense because $(\pi_1F_s/\Gamma_2\pi_1F_s)_s = (H_1F_s)_s \approx H_1F/\Gamma H_1F \approx 0$, since $\pi_1F = \Gamma\pi_1X$ is a perfect π-group. We therefore can assume that F_s is simply connected for all s. The proof of the sufficiency of the conditions in Theorem 1 is now concluded by the observation that if $(F_s)_s$ is a tower of simply connected space such that for all $n \geqslant 0$ $(H_nF_s)_s$ is proisomorphic to some fixed group h_n, then so is the tower π_nF_s. One assumes by induction that for all $j < n$, $(\pi_jF_s)_s$ are, in fact, proisomorphic to some fixed groups π_j. It follows from Lemma 1 that the tower of $(n-1)$-Postnikov section $(P_{n-1}F_s)_s$ is proequivalent to a fixed space B_{n-1}. Now one looks at the fibration

$$(\tilde{F}_s)_s \to (F_s)_s \to B_{n-1}$$

in which $\tilde{F}_s$ are all $(n-1)$-connected. There is an exact sequence of homology

$$\begin{array}{ccccccc} (H_{n+1}F_s)_s & \to & H_{n+1}B_{n-1} & \to & (\pi_n F_s)_s & \to & 0 \\ \downarrow = & & \downarrow = & & \downarrow & & \\ h_{n+1} & \to & H_{n+1}B_{n-1} & \to & H_{n+1}B_{n-1}/h_{n+1} & \to & 0 \end{array}$$

Thus one gets from the five lemmas the inductive assumption for n. Therefore, for all j the progroups $(\pi_j F_s)_s$, and thus $(\pi_j X_s)_s$, are isomorphic to some fixed groups. By Lemma 2.1 $(X_s)_s$ is weakly proequivalent to a fixed space X. By [2], X must be nilpotent. This completes the proof of Theorem 1.

THE HEBREW UNIVERSITY OF JERUSALEM.
YALE UNIVERSITY, NEW HAVEN, CONN.

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