QUILLEN'S WORK ON THE ADAMS CONJECTURE

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Abstract. In the 1960's and 1970's, the Adams Conjecture figured prominently both in homotopy theory and in geometric topology. Quillen sketched one way to attack the conjecture and then proved it with an entirely different line of argument. Both of his approaches led to spectacular and beautiful new mathematics.

1. Background on the Adams Conjecture

For a finite CW-complex X, let $KO(X)$ be the Grothendieck group of finite-dimensional real vector bundles over X, and $J(X)$ the quotient of $KO(X)$ by the subgroup generated by differences $\xi - \eta$, where ξ and η are vector bundles whose associated sphere bundles are fibre-homotopy equivalent. For any integer k, let Ψ^k be the Adams operation on $KO(X)$ constructed in [1]. In [2, 1.2] J. F. Adams made the following proposal.

1.1. **Conjecture.** *If k is an integer, X is a finite CW-complex and $y \in KO(X)$, then there exists a non-negative integer $e = e(k, y)$ such that $k^e(y - \Psi^k y)$ maps to zero in $J(X)$.*

This quickly became known as the *Adams Conjecture*. Let O be the stable orthogonal group and G the monoid of stable self-homotopy equivalences of the sphere, so that the spaces BO and BG classify respectively stable real vector bundles and stable spherical fibrations. There is a natural map $\sigma\colon BO \to BG$ which assigns to each vector bundle its associated sphere bundle. Conjecture 1.1 is equivalent to the statement that for each k the composite map

$$BO \xrightarrow{1-\Psi^k} BO \xrightarrow{\sigma} BG \longrightarrow BG[1/k] \tag{1.2}$$

is null homotopic on finite skeleta.

1.3. *Remark.* There is also a complex form of Conjecture 1.1. Let $K(X)$ be the Grothendieck group of finite-dimensional complex vector bundles over X, and $J'(X)$ the quotient of $K(X)$ obtained as above by identifying two complex vector bundles if their associated sphere

Date: October 20, 2011.

bundles are fibre-homotopy equivalent. Let Ψ^k be the Adams operation on $K(X)$. The complex conjecture asserts that if k and X are as in 1.1 and $y \in K(X)$, then there exists $e = e(k, y)$ such that $k^e(y - \Psi^k y)$ maps to zero in $J'(X)$.

Before Adams. Atiyah had originally defined $J(X)$ in the course of proving that two real vector bundles over X have the same image in $J(X)$ if and only if the bundles have stably equivalent Thom complexes [7]. Adams himself was particularly interested in $X = S^n$: the map $\sigma_\#\colon \pi_n BO \to \pi_n BG$ can be identified with G. W. Whitehead's J-homomorphism [20] $\pi_{n-1}O \to \pi_{n-1}G \cong \pi^s_{n-1}S^0$, and so for $X = S^n$, $J(X)$ is the image of the J-homomorphism. There was differential topology in the mix too. Milnor and Kervaire had observed that an element in the kernel of the J-homomorphism corresponds to a standard sphere with a twisted normal framing which bounds a parallelizable manifold [13], and had noticed that it is very interesting to look at the top Pontrjagin class of the almost parallelizable manifold obtained by capping off that boundary sphere.

Adams' work. In [2], Adams used a direct geometric construction to prove 1.1 for bundles y which can be written as integral linear combinations of one and two dimensional bundles, and devised a clever splitting argument to deduce 1.1 for $X = S^{4s}$, at the cost of an extra factor of 2 if s is even.

Conjecture 1.1 essentially provides an upper bound for the size of $J(X)$. Adams also constructed a lower bound [3] for $J(X)$ and proved that for any X the upper and lower bounds agree [4]. Mysteriously enough, the cannibalistic characteristic classes that contribute to the lower bound also involve the operations Ψ^k (cf. 4.2).

2. Quillen's first approach to the Adams Conjecture

This appears in [14]. The approach is based on etale homotopy theory, which assigns to every reasonable scheme V a space-like object V_{et} called its etale homotopy type. In 1967 V_{et} was a pro-object in the homotopy category of spaces [6], but eventually it was lifted to a pro-object in the category of spaces itself [10]. For our purposes it is enough to take the homotopy limit of this latter pro-object and treat V_{et} as a space.

Quillen's approach is based on the three observations below. Let p be a chosen prime number. Note that up to homotopy the sphere bundle associated to a vector bundle ξ is just the complement $\xi \setminus 0$ of the zero-section.

2.1. *Homotopy data is visible in finite characteristic.* Let R be the strict henselization of $\mathbb{Z}$ at p (concretely, let R be the subring of the maximal unramified extension of $\mathbb{Z}_p$ consisting of elements which are algebraic over $\mathbb{Z}$) and let $k \cong \bar{\mathbb{F}}_p$ be the quotient of R by its unique maximal ideal. Choose an embedding of R in $\mathbb{C}$, so that any scheme V_R over $\operatorname{Spec} R$ gives rise by base change to schemes $V_{\mathbb{C}}$ and V_k. The scheme $V_{\mathbb{C}}$ has a classical homotopy type $V_{\mathbb{C},\mathrm{cl}}$ and under reasonable conditions the comparison theorems of Artin and Mazur [6] give canonical equivalences

$$V\hat{_{\mathbb{C},\mathrm{cl}}} \longrightarrow V\hat{_{\mathbb{C},\mathrm{et}}} \longrightarrow V\hat{_{R,\mathrm{et}}} \longleftarrow V\hat{_{k,\mathrm{et}}}$$

where the superscript "caret" denotes profinite completion away from p. This gives algebraic access over the field k of characteristic p to the completion away from p of a classical homotopy type.

2.2. *Frobenius maps give Adams operations.* If V is a scheme of characteristic p and E is an algebraic vector bundle over V, let $E^{(p)} = \Phi^* E$, where $\Phi\colon V \to V$ is the Frobenius map. Then in $K(V)$ there is an equality

$$\Psi^p(E) = E^{(p)}\,.$$

Here $K(V)$ is the Grothendieck group of algebraic vector bundles over V with relations given by short exact sequences, and the Adams operation Ψ^p on $K(V)$ is defined exactly as Adams defined Ψ^p in [1]. This equality is more or less immediate if E is a line bundle and can be deduced in general from a "splitting principle."

2.3. *Frobenius maps give etale equivalences.* If E is an algebraic vector bundle over a variety in characteristic p, then the Frobenius map on E itself induces a morphism $E \to E^{(p)}$ which preserves the zero section and restricts to a map $E \setminus 0 \to E^{(p)} \setminus 0$. This latter map is purely inseparable and amounts to what Quillen loosely terms a "homeomorphism" in the etale topology; in particular, the map gives an equivalence

$$(E \setminus 0)\hat{_{\mathrm{et}}} \sim (E^{(p)} \setminus 0)\hat{_{\mathrm{et}}}\,.$$

The proposed line of proof. In an ideal world, the complex form (1.3) of the Adams Conjecture could be proved like this. Since the Adams operations are additive and satisfy $\Psi^a\Psi^b = \Psi^{ab}$, it is enough to treat 1.3 when the Adams operation involved is Ψ^p for a prime p. Let $\hat{\mathcal{S}}(X)$ be obtained from the Grothendieck group of spherical fibrations over X by identifying two spherical fibrations if they become fibre homotopy equivalent after profinite completion away from p. Let R and k be as above, and let $E_R \to V_R$ be an algebraic model for the tautological bundle over a Grassmannian. By 2.3 applied to $E_k \to V_k$, there is a fibre homotopy equivalence

$$\begin{array}{ccc} (E_k \setminus 0)\hat{}_{\mathrm{et}} & \longrightarrow & (E_k^{(p)} \setminus 0)\hat{}_{\mathrm{et}} \\ & \searrow \quad \swarrow & \\ & V\hat{}_{k,\mathrm{et}} & \end{array}$$

and so, after deleting zero sections, E_k and $E_k^{(p)}$ give rise to the same element of $\hat{\mathcal{S}}(V\hat{}_{k,\mathrm{et}})$. By 2.2, E_k and $\Psi^p(E_k)$ give rise to the same element of $\hat{\mathcal{S}}(V\hat{}_{k,\mathrm{et}})$. By 2.1, $E_{\mathbb{C}}$ and $\Psi^p(E_{\mathbb{C}})$ give rise to the same element of $\hat{\mathcal{S}}(V\hat{}_{\mathbb{C},\mathrm{cl}})$. But this is a statement about the universal bundle over an ordinary Grassmannian, and easily leads to the Adams Conjecture.

The obstacle. Quillen deals at length with some technicalities arising from the fact that $\Psi^p(E)$ is in general a virtual bundle, but the major problem with the argument above is the implicit assumption that deleting the zero section from an algebraic vector bundle necessarily gives a spherical fibration of etale homotopy types. Quillen's argument thus relied on a conjecture that, under appropriate conditions, if $E \to V$ is an n-dimensional algebraic vector bundle, then $(E \setminus 0)\hat{}_{\mathrm{et}} \to V\hat{}_{\mathrm{et}}$ is naturally a fibration with fibre $\hat{S}^{2n-1}$.

3. Quillen's proof of the Adams Conjecture

Quillen's arguments from [15] are short, and the paper is less than 14 pages long. As above, it is sufficient to prove the Adams Conjecture (1.1) in the special case in which k is a prime number p. The trickiest issues come up when p is odd. This is a consequence of the fact that if $p = 2$, the Adams Conjecture involves localizing *away* from 2; such a localization presents real K-theory as a summand of complex K-theory, and allows 1.1 to be deduced from its complex version (1.3). It turns out that this complex version is slightly easier to handle.

We consider the trickier case. Let p be an odd prime, and let J_p be the composite of the localization map $BG \to BG[1/p]$ with the map $\sigma\colon BO \to BG$ from 1.2. Quillen proves the following statement,

3.1. **Theorem.** *The composite map*

$$J_p(\Psi^p - 1)\colon BO \to BG[1/p]$$

is null homotopic.

The proof breaks into three relatively distinct steps. Let k be the algebraic closure of the finite field $\mathbb{F}_p$, $k_q \subset k$ the finite field with q elements, and $O_n(k_q)$ the finite group of orthogonal matrices in $GL_n(k_q)$.

These are orthogonal matrices taken with respect to the standard quadratic form given by a sum of squares. The infinite orthogonal group $O(k)$ is the discrete group obtained as the union $\cup_{n,q} O_n(k_q)$.

3.2. *Program for proving 3.1.* These are the three steps.

(1) Find a good map $\alpha : BO(k) \to BO$.
(2) Prove that the composite $J_p(\Psi^p - 1)\alpha$ is null homotopic.
(3) Calculate that α induces an isomorphism on $\mathbb{Z}/\ell$ cohomology for any prime $\ell \neq p$.

Theorem 3.1 now follows from obstruction theory, since $BG[1/p]$ is an H-space (in fact, an infinite loop space) with finite homotopy groups.

The good map α comes from an enhancement of Brauer lifting theory. Suppose that $\phi : k^* \to \mathbb{C}^*$ is an embedding and that $\pi : \Gamma \to GL_n(k)$ is a representation of the finite group Γ over k. The modular character χ of π is the complex-valued function on Γ given by

$$\chi(\gamma) = \sum \phi(e_i)$$

where e_i runs through the eigenvalues of $\pi(\gamma)$, with multiplicities taken into account. Brauer theory shows that χ is the character of a (unique) virtual complex representation of Γ; this representation yields an associated virtual vector bundle over $B\Gamma$ and a corresponding homotopy class of maps $B\Gamma \to BU$. Quillen proves that a representation $\Gamma \to O_n(k)$ gives rise in a similar fashion to a map $B\Gamma \to BO$. In particular, the standard inclusions $O_n(k_q) \to O_n(k)$ produce a collection of maps $\alpha_{n,q}\colon BO_n(k_q) \to BO$ which are compatible up to homotopy as n and q vary (a character calculation), and pass in the colimit to a map $\alpha : BO(k) \to BO$.

To prove that $J_p(\Psi^p - 1)\alpha$ is null homotopic, it is enough to show that $J_p(\Psi^p - 1)\alpha_{n,q}$ is null homotopic for each n, q; all $\lim^1$ issues are trivial because the homotopy groups of $BG[1/p]$ are finite. This focuses attention on virtual real representations ξ of a finite group Γ, since $\alpha_{n,q}$ is derived from such a virtual representation of $O_n(k_q)$. Say that *the Adams conjecture is true for* ξ if $J_p(\Psi^p - 1)\alpha_\xi$ is null homotopic, where $\alpha_\xi : B\Gamma \to BO$ is the element of $KO(B\Gamma)$ obtained from ξ. Adams' geometric argument [2] shows that the Adams conjecture is true for ξ if ξ is an integral linear combination of one and two dimensional real representations, and a transfer argument shows that the same is true if ξ is an integral linear combination of representations of Γ induced from one and two dimensional representations of subgroups of Γ. Quillen completes the proof of 3.2(2) with an argument based on Brauer induction to the effect that *every* real representation of a finite group is such an integral linear combination.

At least to the present author's taste, the *pièce de résistance* of the paper is the proof of 3.2(3): simple and fascinating. We will concentrate on $\ell = 2$ (recall that p is odd); the argument for odd $\ell \neq p$ is slightly different in detail but not significantly more complicated. Let $H^* = H^*(-;\mathbb{Z}/2)$, and say that some collection $\{\Gamma_i\}$ of subgroups of a group Γ *detects cohomology* if the restriction map

$$H^*B\Gamma \to \prod_i H^*B\Gamma_i$$

is injective. The trick is to show that the diagonal elementary abelian 2-subgroup $D = \langle \pm 1 \rangle^n \subset O_n(k)$ detects cohomology. Indeed, if this is the case, then there is an injection

$$H^*BO_n(k) \to (H^*BD)^{\Sigma_n} \cong \mathbb{Z}/2[s_1, \ldots, s_n]$$

where Σ_n is the group of standard basis vector permutations normalizing D and s_i is the i'th elementary symmetric function in the fundamental degree 1 cohomology classes of $D = \langle \pm 1 \rangle^n$. But under the composite map

$$H^*BO \xrightarrow{\alpha^*} H^*BO(k) \longrightarrow H^*BO_n(k) \longrightarrow \mathbb{Z}/2[s_1, \ldots, s_n]$$

the i'th Stiefel-Whitney class $w_i \in H^iBO$ restricts essentially by construction to s_i. This gives an isomorphism

$$H^*BO_n(k) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$$

and passing to the limit the desired isomorphism

$$H^*BO(k) \cong Z/2[w_1, w_2, \ldots] \cong H^*BO\,.$$

What remains is to prove that D detects cohomology in $O_n(k)$. An algebraic argument shows that every elementary abelian 2-subgroup of $O_n(k)$ is conjugate to a subgroup of D, so it is sufficient to show that the family of elementary abelian 2-subgroups detects cohomology in $O_n(k)$, or even that there exists some cofinal collection of q's such that elementary abelian 2-subgroups detect cohomology in each $O_n(k_q)$.

Consider the groups $O_n(k_q)$ for q's which are congruent to 1 mod 4. Write $n = 2m + e$ with e equal 0 or 1, and let

$$N \cong (\Sigma_m \wr O_2(k_q)) \times \langle \pm 1 \rangle^e$$

be the normalizer in $O_n(k_q)$ of the largest possible subgroup of block diagonal 2×2 matrices. By counting, the index of N in $O_n(k_q)$ is odd, so N detects cohomology in $O_n(k_q)$. A direct calculation with dihedral groups (Quillen invokes [16] at one point but then relents and works it out!) shows that elementary abelian 2-groups detect cohomology in

$O_2(k_q)$. All that is needed to tie these two facts together is a way to pass from $O_2(k_q)$ to $\Sigma_m \wr O_2(k_q)$. The following lemma does just that.

3.3. **Lemma.** [15, 3.4] *Let* Γ *be a group whose mod* ℓ *cohomology is detected by a family of abelian subgroups of exponent dividing* ℓ^a *with* $a \geq 1$. *Then* $\Sigma_n \wr \Gamma$ *has the same property.*

Quillen proves 3.3 with an argument [15, 3.1] that refers to Smith theory and the Serre spectral sequence.

3.4. *Remark.* The same argument with minor adjustments shows there is a map $BGL(k) \to BU$ which induces an isomorphism on cohomology with mod ℓ coefficients, for all primes $\ell \neq p$. This gives a proof of 1.3.

Reading Quillen's paper is a breathtaking experience, even at this distance in time.

4. Aftermath

There have been at least three other proofs of the Adams Conjecture. Friedlander [9] completed Quillen's initial sketch proof. Sullivan [19] gave a proof which used etale homotopy theory but exploited discontinuous automorphisms of $\mathbb{C}$ over $\mathbb{Q}$ rather than Frobenius maps in finite characteristic. Later on Becker and Gottlieb [8] found a remarkably simple proof that hinged on the transfer map for fibre bundles.

4.1. *Algebraic* K*-theory.* Quillen himself extended the ideas from [14] and [15] (Brauer lifting, Ψ^p vs. Frobenius, detecting cohomology on wreath products...) to determine $H^*(BGL_n(k); \mathbb{Z}/\ell)$ for any finite field k of characteristic p and any prime $\ell \neq p$ [17]; he also went a bit further and showed that the reduced cohomology $\tilde{H}^*(BGL_n(k); \mathbb{Z}/p)$ vanishes in a stable range. Let q be the cardinality of k and $F\Psi^q$ the homotopy fibre of the endomorphism $\Psi^q - 1$ of BU. In the limit his argument gives a map

$$BGL(k) \to F\Psi^q$$

which induces an isomorphism on integral cohomology. Of course this map is not a weak homotopy equivalence, if only because the source has a large nonabelian fundamental group, but the target does not. Improbably enough, Quillen saw an opportunity here. He handcrafted the plus construction, observed that $BGL(k)^+ \to F\Psi^q$ *is* a weak homotopy equivalence, and proposed that for any ring R the higher algebraic K-groups $K_i(R)$, $i > 0$, should be defined by $K_i(R) = \pi_i BGL(R)^+$. In this way he invented the present-day notion of higher algebraic K-theory and provided the first complete calculations. Namely, for k as above,

$$K_i(k) \cong \pi_i F\Psi^q \cong \begin{cases} \mathbb{Z}/(q^j - 1) & i > 0,\, i = 2j-1 \\ 0 & i > 0,\, i \text{ even} \end{cases} .$$

Adams has an intriguing expository account of the plus construction and related subjects in [5, §3].

4.2. *Upper and lower.* But now back to Adams' paper. Again, let k be a finite field with q elements and $F\Psi^q$ the homotopy fibre of the endomorphism $\Psi^q - 1$ of BU; let SG be the identity component of G. Quillen's solution in [15] of the complex Adams Conjecture (1.3) gives the following commutative diagram, in which the rows are fibration sequences and the right vertical map corresponds to the complex J homomorphism.

$$\begin{array}{ccccc} F\Psi^q & \longrightarrow & BU & \xrightarrow{\Psi^q - 1} & BU \\ \downarrow & & \downarrow & & \downarrow \\ SG[1/q] & \longrightarrow & * & \longrightarrow & BSG[1/q] \end{array}$$

If interest lies in ℓ-primary information for an odd prime ℓ and q reduces to a generator of $(\mathbb{Z}/\ell^2)^\times$, the space $F\Psi^q$ embodies Adams' upper bound for $J(–)$. Quillen's interpretation of $F\Psi^q$ in terms of the algebraic K-theory of k gives a map $\epsilon\colon SG \to F\Psi^q$. In a letter to Milnor [18], Quillen interprets this map as (the negative of) Adams' e-invariant, or, essentially, as Adams' *lower* bound for $J(–)$. For q as above, the composite

$$F\Psi^q \longrightarrow SG[1/q] \xrightarrow{\epsilon[1/q]} (F\Psi^q)[1/q] \sim F\Psi^q$$

is an equivalence at ℓ [18]. From this point of view, Quillen gave geometric interpretations of both sides of Adams' work, using the same space on each side. For references, details, related work, and an interpretation in terms of discrete groups see [12, p. 478ff] and [11, §5].

A personal note. I was a graduate student at MIT in the early 1970's, when Quillen was working on the Adams Conjecture, algebraic K-theory, equivariant cohomology rings, and formal group laws. What an inspiration he was! So many of us graduate students wanted to be just like him, to lecture so gracefully, to write so persuasively, and to have such shockingly beautiful ideas. Being around him was a privilege that meant a lot to me. Thanks, Dan.

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