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Self homotopy equivalences of virtually nilpotent spaces*

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§1. Introduction

The aim of this paper is to prove Theorem 1.1 below, a generalization to *virtually nilpotent spaces* of a result of Wilkerson [13] and Sullivan [12]. We recall that a CW complex Y is *virtually nilpotent* if

(i) Y is connected,

(ii) $\pi_1 Y$ is *virtually nilpotent* (i.e. has a nilpotent subgroup of finite index) and

(iii) for every integer $n > 1$, $\pi_1 Y$ has a subgroup of finite index which acts nilpotently on $\pi_n Y$. The class of virtually nilpotent spaces is much larger than the class of nilpotent spaces. For instance such non-nilpotent spaces as the Klein bottle and the real projective spaces are virtually nilpotent, and so is, of course, any connected space with a finite fundamental group.

1.1. THEOREM. *Let Y be a virtually nilpotent finite* CW *complex. Then the classifying space of the topological monoid of the self homotopy equivalences of Y is of finite type* (i.e. has the homotopy type of a CW complex with a finite number of cells in each dimension). *In fact it has the somewhat stronger property that each of its homotopy groups is of finite type* (i.e. has a classifying space of finite type).

1.2. *Remark.* For abelian groups, being of finite type is the same as being finitely generated, but for non-abelian groups, being of finite type is stronger than being finitely generated or even being finitely presented.

1.3. *Remark.* As it is easy to verify that, in Theorem 1.1, the higher homotopy groups in question are finitely generated, the main content of Theorem 1.1 is that *the group of homotopy classes of self homotopy equivalences of Y is of finite type.*

1.4. ORGANIZATION OF THE PAPER. The paper consists essentially of three parts:

(i) After a brief discussion (in §2) of the notion of *finite type* for groups and simplicial sets, we reduce Theorem 1.1 (in §3) to a similar (and in fact equivalent)

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statement (3.6) about the *homotopy automorphism complex* (i.e. complex of self loop homotopy equivalences) *of a simplicial group* and then (in §4) to a similar statement (4.1) about the *homotopy automorphism complex of a simplicial virtually nilpotent group*. The arguments are standard, except in 3.5, where we make the transition from pointed simplicial sets to simplicial groups and have to show that the usual notion of a function complex of maps between two simplicial groups is indeed the "correct" one.

(ii) In the next two sections we make the crucial transition from *homotopy automorphisms* to *automorphisms*, i.e. we reduce Theorem 4.1 to a similar statement (6.1) about the *automorphism complex of a simplicial virtually nilpotent group*. A key step in the argument is a curious lemma (5.1) which states that, *under suitable circumstances, the homotopy groups of the automorphism complex of a simplicial module differ by only a finite amount from the homotopy groups of the homotopy automorphism complex.*

(iii) The last two sections are devoted to a proof of Theorem 6.1. It turns out that it suffices to show that the automorphism complexes involved are dimension wise of finite type and this we then do by combining variations on arguments of Baumslags proof that the automorphism group of a finitely generated virtually nilpotent group is finitely presented [1, Ch. 4] with the result of Borel and Serre that arithmetic subgroups of algebraic groups are of finite type [2, §11]. Of course it would be nice if one could do this without resorting to such non-homotopical notions as algebraic groups and their arithmetic subgroups.

§2. Finite type

We start with a brief review of the notions of finite type for simplicial sets and for groups, and note in particular (2.9) that *a connected simplicial set with finitely generated higher homotopy groups is of finite type if and only if its fundamental group is of finite type.*

2.1. SIMPLICAL SETS OF FINITE TYPE. A simplicial set X is said to be of *finite type* if, for every integer $n \geq 0$, there exists a map $f_n : F_n \to X$ such that

(i) F_n is finite (i.e. has only a finite number of non-degenerate simplices) and

(ii) f_n induces, for every vertex $v \in F_n$ and every integer $0 \leq i \leq n$, an isomorphism $\pi_i(F_n; v) \approx \pi_i(X; f_n v)$.

This definition readily implies ([8], [11, Ch. III and Ch. VI]):

2.2. PROPOSITION. *A simplicial set X is of finite type if and only if its realization $|X|$ has the homotopy type of a* CW *complex with a finite number of cells in each dimension.*

2.3. PROPOSITION. *A reduced* (i.e. only one vertex) *simplicial set X is of finite type if and only if its simplicial loop group GX has the loop homotopy type of a free simplicial group which is finitely generated in each dimension.*

The next two propositions are very useful ones.

2.4. PROPOSITION. *Let U be a bisimplicial set such that, for every integer $k \geq 0$, the simplicial set $U_{k,*}$ is of finite type. Then the diagonal diag U is also of finite type.*

2.5. PROPOSITION. *Let $p: E \to B$ be a fibration onto and assume that all its fibres are of finite type. Then E is of finite type if and only if B is so.*

Proofs. The proof of 2.4 is easy once the diagonal of the bisimplicial set U has been identified with the "realization" of U [3, Ch. XII, 3.4]. The "if" part of 2.5 is straightforward. To prove the "only if" part of 2.5, let U and V be the bisimplicial sets such that, for every integer $k \geq 0$,

$$V_{k,*} = B \quad \text{and} \quad U_{k,*} = E \times_B \cdots \times_B E \ (k+1 \text{ factors})$$

and let $U \to V$ be the obvious map. Then it is not hard to verify that, for every integer $n \geq 0$, the induced map $U_{*,n} \to V_{*,n} = B_n$ is a weak homotopy equivalence and so is therefore [3, p. 335] the induced map diag $U \to$ diag $V = B$. The desired result now follows from 2.4, the "if" part of 2.5 and the fact that $U_{0,*} = E$ and that, for every integer $k \geq 0$, the face maps $d_i: U_{k+1,*} \to U_{k,*}$ are fibrations with the fibres of p as fibres.

Next we consider

2.6. GROUPS OF FINITE TYPE. A group G is said to be of *finite type* if the simplicial set $K(G, 1)$ is of finite type.

2.7. EXAMPLES. Using 2.5 one readily verifies that the following groups are of finite type:

(i) all *finitely generated free groups,*

(ii) all *finite groups*

(iii) all *finitely generated abelian groups,*

(iv) all *finitely generated nilpotent groups,*

(v) all *finitely generated virtually nilpotent* (see § 1) *groups,* and

(vi) all *homotopy groups of a simplicial set which is virtually nilpotent* (see § 1) *and of finite type.*

Less obvious are

2.8. EXAMPLES. (i) Every *arithmetic subgroup of an algebraic group* is of finite type. This is a result of Borel–Serre [2, § 11].

(ii) The *group of automorphisms of a finitely generated virtually nilpotent group* is of finite type. To prove this one combines Baumslag's proof of [1, th. 4.7] with 2.5 and 2.8 (i).

We end with several propositions which will be needed later.

2.9. PROPOSITION. *Let X be a connected simplicial set and assume that $\pi_n X$ is of finite type for $n > 1$. Then $\pi_1 X$ is of finite type if and only if X is of finite type.*

2.10. PROPOSITION. *Let C be a simplicial group such that C_n is of finite type for all $n \geq 0$. Then its classifying complex $\bar{W}C$* [11, Ch. IV] *is of finite type.*

2.11. PROPOSITION. *Let $G \to \{G_i\}$ be a pro-isomorphism of groups* [2, Ch. III] *in which each G_i is of finite type. Then G is also of finite type.*

Proofs. Propositions 2.9 and 2.10 follow readily from Propositions 2.5 and 2.4 respectively, while Proposition 2.11 is an immediate consequence of the fact that any retract of a simplicial set of finite type is also of finite type.

§ 3. Reduction to simplicial groups

In this section we reduce Theorem 1.1 to similar and equivalent results for *simplicial sets* (3.2), *reduced* (i.e. only one vertex) *simplicial sets* (3.4) and *simplicial groups* (3.6). Most of the arguments are routine. However, in the last reduction one runs into the problem that the loop group functor G is not a simplicial functor with respect to the usual simplicial structures on the categories of reduced simplicial sets and simplicial groups. To get around this difficulty we introduce on the category of reduced simplicial sets a new simplicial structure which is better behaved with respect to the functor G and which gives rise to function complexes homotopically equivalent to the usual ones. Of course one could instead have appealed to the rather general Proposition 5.4 of [5].

3.1. REDUCTION TO SIMPLICIAL SETS. For a CW complex Y let *haut* Y denote its *simplicial monoid of self homotopy equivalences*, i.e. the simplicial monoid which has as its n-simplices the homotopy equivalences $|\Delta[n]| \times Y \to Y$, and for a fibrant (i.e. satisfying the extension condition [11, § 1]) simplicial set X, let haut X denote its simplicial monoid of homotopy automorphisms, i.e. the simplicial monoid which has as its n-simplices the weak homotopy equivalences $\Delta[n]\times X \to X$. Using

(i) the adjointness of the realization functor $| \ |$ and the singular functor Sin,

(ii) the fact that, for every CW complex Y and fibrant simplicial set X, the adjunction maps $|\text{Sin}\, Y| \to Y$ and $X \to \text{Sin}|X|$ are homotopy equivalences, and

(iii) the fact that, for every simplicial set X, the obvious maps $|\Delta[n]\times X| \to |\Delta[n]|\times|X|$ are homeomorphisms, one readily verifies that *the induced maps* π_n haut $X \to \pi_n$ haut $|X|$ *are isomorphisms for all* $n \geq 0$. As haut Y is clearly isomorphic to the singular complex of the topological monoid of self homotopy equivalences of Y, if follows that Theorem 1.1 is equivalent to

3.2. THEOREM. *Let X be a virtually nilpotent fibrant simplicial set which has the (weak) homotopy type of a finite simplicial set. Then π_n haut X is of finite type for all $n \geq 0$.*

3.3. REDUCTION TO REDUCED SIMPLICIAL SETS. For a reduced (i.e. only one vertex) fibrant simplicial set K, denote by $\text{haut}_* K$ the submonoid of haut K which "keeps the vertex fixed" and note that there is an obvious fibration haut $K \to K$ with $\text{haut}_* K$ as fibre. Using 2.5 and 2.7 (vi) one then readily sees that Theorem 3.2 is equivalent to

3.4. THEOREM. *Let K be a virtually nilpotent fibrant simplicial set which is reduced and has the (weak) homotopy type of a finite simplicial set. Then $\pi_n\, \text{haut}_* K$ is of finite type for all $n \geq 0$.*

3.5. REDUCTION TO SIMPLICIAL GROUPS. We start with constructing a new simplicial structure on the category of reduced simplicial sets along the lines of [7, § 12], i.e. for a simplicial set X and a reduced simplicial set K, we denote by $X \cdot K$ the reduced simplicial set which is the quotient of $X \times K$ by the equivalence relation: $(x_1, k_1) \sim (x_2, k_2)$ *if and only if* $k_1 = k_2 = s_0^i k$ *for some non-degenerate* $k \in K$ *and* $d_0^{i+1} x_1 = d_0^{i+1} x_2$, and note that this definition readily implies the existence of a *natural isomorphism* $(X' \times X) \cdot K \approx X' \cdot (X \cdot K)$.

Next, for a fibrant reduced simplicial set K, denote by $\text{haut}_0 K \subset \text{haut}_* K$ the subcomplex consisting of the maps $\Delta[n]\times K \to K$ which factor through $\Delta[n] \cdot K$. Then $\text{haut}_0 K$ *is clearly a submonoid of* $\text{haut}_* K$. Moreover the usual retraction $\Delta[1]\times\Delta[n] \to \Delta[n]$ of $\Delta[n]$ onto its first (or last) vertex induces a retraction of $\Delta[n] \cdot K$ onto K and hence [5, § 6] *the induced maps* $\pi_n\, \text{haut}_0 K \to \pi_n\, \text{haut}_* K$ *are isomorphisms for all* $n \geq 0$.

Finally, for a simplicial group C, denote by haut C its *simplicial monoid of homotopy automorphisms*, i.e. the simplicial monoid which has as its n-simplices the homomorphisms $\Delta[n]\otimes C \to C$ which are weak (loop homotopy) equivalences [11, Ch. VI] (i.e. induce isomorphisms on π_n for all $n \geq 0$). Using [11, Ch. VI],

(i) the adjointness of the loop group functor G and the classifying complex functor $\bar{W}$,

(ii) the fact that, for every fibrant reduced simplicial set K and every free simplicial group C, the adjunction maps $K \to \bar{W}GK$ and $G\bar{W}C \to C$ are respectively a homotopy equivalence and a loop homotopy equivalence, and

(iii) the fact that, for every reduced simplicial set K, the homomorphisms $\Delta[n]\otimes GK \to G(\Delta[n]\cdot K)$, given by [7, p. 118] $(x, \tau k) \to \tau(s_0x, k)$, are actually isomorphisms, one verifies that *the induced maps* π_n $\text{haut}_0\, K \to \pi_n$ haut GK *are isomorphisms for all* $n \geq 0$. It now follows that Theorem 3.4 is equivalent to

3.6. THEOREM. *Let C be a free simplicial group which is finitely generated* (i.e. has a finite number of non-degenerate generators) *and has a virtually nilpotent classifying complex* $\bar{W}C$ [11, Ch. IV]. *Then* π_n haut C *is of finite type for all* $n \geq 0$.

§ 4. Reduction to simplicial virtually nilpotent groups

Now we reduce Theorem 3.6 to a similar result for simplicial virtually nilpotent groups (4.1). To state this result denote, for a homomorphism of simplicial groups $C \to \pi$, by $\text{haut}_\pi\, C \subset \text{haut}\, C$ the *simplicial monoid of homotopy automorphisms of C over* π, i.e. the simplicial monoid which has as its n-simplices the commutative diagrams

$$\begin{array}{ccc} \Delta[n]\otimes C & \to & C \\ & \searrow \quad \swarrow & \\ & \pi & \end{array}$$

in which the top map is in haut C and the other maps are the obvious ones. Furthermore, for a (simplicial) group B, let $\Gamma_i B$ be the i-th term of its *lower central series* (i.e. $\Gamma_1 B = B$ and $\Gamma_i B = [\Gamma_{i-1}B, B]$ for $i > 1$. Then one has

4.1. THEOREM. *Let* $1 \to B \to C \to \pi \to 1$ *be an exact sequence of simplicial groups such that*

(i) π *is discrete and finite,*

(ii) *C is free and finitely generated* (see 3.6), *and*

(iii) *the classifying complex* $\bar{W}B$ [11, Ch. IV] *is nilpotent.*
Then the groups π_n haut_π $C/\Gamma_i B$ $(n \geq 0, i \geq 1)$ *are of finite type.*

That indeed this Theorem 4.1 implies Theorem 3.6 is an immediate consequence of 2.11 and the following three propositions.

4.2. PROPOSITION. *Let* $1 \to B \to C \to \pi \to 1$ *be as in* 4.1. *Then the obvious maps*

$$\pi_n \text{ haut}_\pi C \to \{\pi_n \text{ haut}_\pi C/\Gamma_i B\} \qquad n \geq 0$$

are pro-isomorphisms of groups [3, Ch. III].

Proof. In view of [3, Ch. III] the obvious map $C \to \{C/\Gamma_i B\}$ is a weak pro-homotopy equivalence and it is not difficult to show, using induction on the number of non-degenerate generators of C, that so is the induced map of function complexes over π

$$\text{hom}_\pi (C, C) \to \{\text{hom}_\pi (C, C/\Gamma_i B)\}$$

and the desired result now follows readily from the obvious isomorphisms

$$\text{hom}_\pi (C, C/\Gamma_i B) \approx \text{hom}_\pi (C/\Gamma_i B, C/\Gamma_i B)$$

4.3. PROPOSITION. *Let* $C \to \pi$ *be a homomorphism of simplicial groups such that* $\pi_0 C$ *is finitely generated and* π *is discrete and finite. Then* π_n $\text{haut}_\pi C = \pi_n$ $\text{haut}\, C$ *for* $n \geq 1$ *and* π_0 $\text{haut}_\pi C$ *is a subgroup of finite index of* π_0 $\text{haut}\, C$.

Proof. This follows readily from the fact that a finitely generated group (such as $\pi_0 C$) has only a finite number of subgroups of a given finite index.

4.4. PROPOSITION. *Let* C *be a finitely generated free simplicial group with a virtually nilpotent classifying complex* $\bar{W}C$. *Then there exists an exact sequence* $1 \to B \to C \to \pi \to 1$ *of simplicial groups such that*

(i) π *is discrete and finite, and*

(ii) $\bar{W}B$ *is nilpotent.*

Proof. In view of [8] if suffices to show that *every virtually nilpotent finite CW complex* Y *has a nilpotent finite cover.* To prove this let $\varphi \subset \pi_1 Y$ be a nilpotent subgroup of finite index which acts nilpotently on $\pi_n Y$ for $2 \leq n \leq \dim Y$. Then φ acts on the universal cover $\tilde{Y}$ of Y. As φ acts nilpotently on $\pi_n \tilde{Y}$ for $2 \leq n \leq \dim Y$ and $\dim \tilde{Y} = \dim Y$, it follows that φ acts nilpotently on $H_n \tilde{Y}$ for all $n \geq 0$ and therefore on $\pi_n \tilde{Y}$ for all $n \geq 2$. The desired result is now immediate.

§ 5. Automorphisms of simplicial modules

In preparation for the next step in our reduction (in § 6) we prove here a lemma for simplicial modules (5.1) which seems to be of interest in its own right.

For a simplicial π-module M, let $\text{haut}_\pi M$ be its *simplicial monoid of homotopy automorphisms* (an n-simplex of which is a π-module homomorphism $\Delta[n]\otimes M \to M$ which is a homotopy equivalence) and let $\text{aut}_\pi M \subset \text{haut}_\pi M$ be its maximal *simplicial subgroup of automorphisms.* Then one has:

5.1. LEMMA. *Let π be a finite group and let M be a finitely generated* (3.6) *simplicial π-module which, in each dimension, is torsion free as an abelian group. Then the obvious maps $\pi_n \text{aut}_\pi M \to \pi_n \text{haut}_\pi M$ $(n \geq 0)$ have finite kernels and cokernels.*

5.2. *Remark.* Lemma 5.1 remains true if M is *not* required to be torsion free in each dimension, but we don't need this extra generality.

Proof. The proof consists of three parts and will often, explicitly or implicitly, use the fact that [11, Ch. V] *there exists an isomorphism of categories N between the category of simplicial π-modules and the category of differential graded π-modules which are trivial in negative dimensions.* First we note that 5.1 holds if $NM_i = 0$ for $i \neq n, n+1$ and $\pi_i M = 0$ for $i \neq n$. Next we consider a finite direct sum of such simplicial π-modules and finally we treat the general case.

I. Assume that M is as in 5.1 and that in addition $NM_i = 0$ for $i \neq n, n+1$ and $\pi_i M = 0$ for $i \neq n$. Then the boundary map $\partial: NM_{n+1} \to NM_n$ is a monomorphism and we can therefore consider NM_{n+1} as a submodule of NM_n. Furthermore $N(\Delta[K]\otimes M)_n$ is a direct sum of copies of NM_n indexed by the n-simplices of $\Delta[k]$ and if, for every n-simplex $p \in \Delta[k]$ and element $x \in NM_n$, we denote by $x_p \in N(\Delta[k]\otimes M)_n$ the copy of x that lies in the summand indexed by p, then a straightforward calculation yields that the image of the boundary map $\partial: N(\Delta[k]\otimes M)_{n+1} \to N(\Delta[k]\otimes M)_n$ is generated by the elements

$$x_p - x_q \quad \text{where} \quad x \in NM_n \quad \text{and} \quad p, q \in \Delta[k]_n$$

$$x_p \quad \text{where} \quad x \in NM_{n+1} \quad \text{and} \quad p \in \Delta[k]_n$$

Next one notes that a k-simplex $f \in \hom_\pi (M, M)$ is completely determined by a collection of π-modules maps $f_p: NM_n \to NM_n$ indexed by the n-simplices of $\Delta[k]$ and it is not difficult to verify that conversely such a collection $\{f_p\}$

(i) comes from a k-simplex of $\hom_\pi (M, M)$ iff each f_p maps NM_{n+1} into itself and all f_p induce the same endomorphism of NM_n/NM_{n+1},

(ii) comes from a k-simplex of $\mathrm{haut}_\pi M$ iff, in addition to the conditions of (i), the f_p induce an automorphism of NM_n/NM_{n+1} and

(iii) comes from a k-simplex of $\mathrm{aut}_\pi M$ iff, is addition to the conditions of (i) and (ii), each f_p is an automorphism of NM_n.

From this, together with the usual combinatorial formulas for the homotopy groups of a complex satisfying the extension condition [10, p. 5], it is not hard to deduce that *the homotopy groups of* $\mathrm{aut}_\pi M$ *and* $\mathrm{haut}_\pi M$ *vanish in dimension* >0 *and that the map* $\pi_0 \mathrm{aut}_\pi M \to \pi_0 \mathrm{haut}_\pi M$ *can be identified with the inclusion, into the group of automorphisms of* NM_n/NM_{n+1} *that lift to endomorphisms of* NM_n, *of those automorphisms of* NM_n/NM_{n+1} *that lift to automorphisms of* NM_n. To see that this inclusion is of finite index, one notes that NM_n determines an element in the finite group $\mathrm{Ext}_\pi(NM_n/NM_{n+1}, NM_{n+1})$ and that the automorphisms of NM_n/NM_{n+1} that stabilize this element are contained in the image of the inclusion in question.

II. Assume that $M \approx M^0 \oplus \cdots \oplus M^r$ where each M^n $(0 \le n \le r)$ is as in I; in particular $NM_i^n = 0$ if $i \ne n, n+1$ and $\pi_i M^n = 0$ if $i \ne n$. Then one readily verifies, using the functor N, that $\mathrm{hom}_\pi(M^i, M^j) = 0$ for $i > j$. Hence *the* k*-simplices of* $\mathrm{aut}_\pi M$ (*resp.* $\mathrm{haut}_\pi M$) *are in* 1–1 *correspondence with* $(n \times n)$*-matrices* $\{s_{i,j}\}$ (*resp.* $\{t_{i,j}\}$) *with*

$$
\begin{array}{lll}
s_{i,j}, t_{i,j} \in \mathrm{hom}_\pi(M^i, M^j)_k & \text{for} & i < j. \\
s_{ij} = t_{i,j} = 0 & \text{for} & i > j \\
s_{i,i} \in (\mathrm{aut}_\pi M^i)_k & t_{i,i} \in (\mathrm{haut}_\pi M^i)_k &
\end{array}
$$

and the desired result is now immediate.

III. Finally assume that M is merely as in 5.1. For every integer $k \ge -1$, let $E_k M \subset M$ be the maximal simplicial submodule which is trivial in dimensions $\le k$. As M is finitely generated there is an integer r such that $E_r M = 0$ and the finiteness of π now readily implies the existence of an isomorphism of simplicial π-modules

$$
M \otimes Q \approx (E_{-1}M/E_0 M \otimes Q) \oplus \cdots \oplus (E_{r-1}M/E_r M \otimes Q)
$$

Using the functor N it is not hard to show that there is a sequence of positive integers $t = (t_0, \ldots, t_r)$ with t_{k-1} dividing t_k $(1 \le k \le r)$ such that, for the simplicial submodule $t^{-1}M \subset M \otimes Q$ given by

$$
x \in t^{-1}M \quad \textit{iff} \quad t_k x \in E_k M + E_{k-1}(M \otimes Q) \quad \textit{for all} \quad 0 \le k \le r
$$

there is an isomorphism of simplicial modules

$$t^{-1}M \approx E_{-1}t^{-1}M/E_0t^{-1}M \oplus \cdots \oplus E_{r-1}t^{-1}M/E_rt^{-1}M$$

i.e. $t^{-1}M$ *satisfies the conditions of* II. Moreover the naturality of the construction t^{-1} implies the existence of a commutative diagram

$$\begin{array}{ccc} \mathrm{aut}_\pi M & \longrightarrow & \mathrm{aut}_\pi t^{-1}M \\ \downarrow \scriptstyle\mathrm{incl.} & & \downarrow \scriptstyle\mathrm{incl.} \\ \mathrm{haut}_\pi M & \longrightarrow & \mathrm{haut}_\pi t^{-1}M \end{array}$$

in which, because M was assumed to be dimension wise torsion free as an abelian group, the horizontal maps are 1–1 and it remains to show that the maps they induce on the homotopy groups have finite kernels and cokernels in all dimensions ≥ 0.

To do this for the top map we note that a k-simplex $f \in \mathrm{aut}_\pi M$ (resp. $\mathrm{aut}_\pi t^{-1}M$) is completely determined by a collection of automorphisms $f_p : M_{\mathrm{dim}\, p} \to M_{\mathrm{dim}\, p}$ (resp. $t^{-1}M_{\mathrm{dim}\, p} \to t^{-1}M_{\mathrm{dim}\, p}$) indexed by the simplices $p \in \Delta[k]$ of dimension $\leq r+k$. From this and the fact that, in each dimension, M has index in $t^{-1}M$, it is not difficult to deduce that the image of $\mathrm{aut}_\pi M$ in $\mathrm{aut}_\pi t^{-1}M$ also has finite index in each dimension and the desired result follows.

The bottom map is the restriction to certain components of a homomorphism of simplicial abelian groups $\mathrm{hom}_\pi (M, M) \to \mathrm{hom}_\pi (t^{-1}M, t^{-1}M)$ which is 1–1 and has a finite cokernel in each dimension. Hence the induced map $\pi_n\, \mathrm{haut}_\pi M \to \pi_n\, \mathrm{haut}_\pi t^{-1}M$ has a finite kernel for $n \geq 0$ and a finite cokernel for $n > 0$. That this map also has a finite cokernel for $n = 0$ follows from the fact that (see above) the composition $\pi_0\, \mathrm{aut}_\pi M \to \pi_0\, \mathrm{haut}_\pi M \to \pi_0\, \mathrm{haut}_\pi t^{-1}M$ does.

§6. Reduction to automorphisms

The next reduction step is to show that Theorem 4.1 is equivalent to a similar result for the maximal *simplicial* sub-*groups of automorphisms* $\mathrm{aut}_\pi C/\Gamma_iB \subset \mathrm{haut}_\pi C/\Gamma_iB$, i.e.

6.1. THEOREM. $1 \to B \to C \to \pi \to 1$ *be as in* 4.1. *Then the groups* $\pi_n\, \mathrm{aut}_\pi C/\Gamma_iB$ $(n \geq 0, i \geq 1)$ *are of finite type.*

This equivalence follows immediately from

6.2. LEMMA. *Let* $1 \to B \to C \to \pi \to 1$ *be as in* 4.1 (i) *and* (ii). *Then the obvious maps* $\pi_n\ \mathrm{aut}_\pi\ C/\Gamma_i B \to \pi_n\ \mathrm{haut}_\pi\ C/\Gamma_i B$ $(n \geq 0, i \geq 1)$ *have finite kernels and cokernels.*

Proof. Note that there is a pull back diagram

$$\begin{array}{ccc} \mathrm{aut}_\pi\ C/\Gamma_i B & \longrightarrow & \mathrm{haut}_\pi\ C/\Gamma_i B \\ \downarrow & & \downarrow \\ \mathrm{aut}_\pi\ C/\Gamma_2 B & \longrightarrow & \mathrm{haut}_\pi\ C/\Gamma_2 B \end{array}$$

in which, since C is free, the map on the right is a fibration. Hence it suffices to prove the lemma for $i = 2$ only, which we will do by reducing this case to Lemma 5.1.

Consider the commutative diagram

$$\begin{array}{ccccccccc} Z^1(\pi; B/\Gamma_2 B) & \xrightarrow{a} & \mathrm{aut}_\pi\ C/\Gamma_2 B & \xrightarrow{b} & \mathrm{aut}_\pi\ B/\Gamma_2 B & & \xrightarrow{c} & & H^2(\pi; B/\Gamma_2 B) \\ \downarrow \approx & & \downarrow & & \downarrow & & & & \downarrow \approx \\ Z^1(\pi; B/\Gamma_2 B) & \xrightarrow{a} & \mathrm{haut}_\pi\ C/\Gamma_2 B & \xrightarrow{b} & \mathrm{haut}_\pi\ B/\Gamma_2 B & \to & \mathrm{end}_\pi\ B/\Gamma_2 B & \xrightarrow{c} & H^2(\pi; B/\Gamma_2 B) \end{array}$$

constructed as follows:

(i) The maps b are induced by the functor which, to every epimorphism $H \to \pi$ with abelian kernel, assigns this kernel (as a π-module).

(ii) For every 1-cocycle $z \in Z^1(\pi; B_n/\Gamma_2 B_n)$ (i.e. function $z : \pi \to B_n/\Gamma_2 B_n$ such that $z(xy) = xz(y) + z(x)$ for all $x, y \in \pi$), the map $az : \Delta[n] \otimes C/\Gamma_2 B \to C/\Gamma_2 B$ assigns to k-simplices $p \in \Delta[n]$ and $q \in C/\Gamma_2 B$, the k-simplex $p'(zq') \cdot q \in C/\Gamma_2 B$, where q' denotes the image of q in π and p' is the simplicial operator such that $p = p' i_n$, where $i_n \in \Delta[n]$ is the non-degenerate n-simplex.

(iii) For an n-simplex $r \in \mathrm{end}_\pi\ B/\Gamma_2 B = \mathrm{hom}_\pi\ (B/\Gamma_2 B, B/\Gamma_2 B)$ we put $cr = k_n - r'_* k_n$, where $k_n \in H^2(\pi; B_n/\Gamma_2 B_n)$ is the extension class [8, Ch. IV] of $B_n/\Gamma_2 B_n \to C_n/\Gamma_2 B_n \to \pi$ and $r' : B_n/\Gamma_2 B_n \to B_n/\Gamma_2 B_n$ is the restriction of r to the non-degenerate n-simplex $i_n \in \Delta[n]$.

Then it is not difficult to verify that the maps a are 1–1, that $Z^1(\pi; B/\Gamma_2 B)$ acts principally on $\mathrm{aut}_\pi\ C/\Gamma_2 B$ and $\mathrm{haut}_\pi\ C/\Gamma_2 B$ and that the maps b map the resulting quotients isomorphically onto the subcomplexes of $\mathrm{aut}_\pi\ B/\Gamma_2 B$ and $\mathrm{haut}_\pi\ B/\Gamma_2 B$ which go to 0 under c. Moreover the first of these quotients is a simplicial group which acts principally on $\text{aut}_\pi B/\Gamma_2B$ and this readily implies that the image of $\text{aut}_\pi B/\Gamma_2B$ in $H^2(\pi; B/\Gamma_2B)$ is fibrant; as $H^2(\pi; B/\Gamma_2B)$ is dimension wise finite, so is this image and its homotopy groups are thus finite in all dimensions ≥ 0. To obtain a similar result for the image of $\text{haut}_\pi B/\Gamma_2B$ in $H^2(\pi; B/\Gamma_2B)$, one notes that the map $c: \text{end}_\pi B/\Gamma_2B \to H^2(\pi; B/\Gamma_2B)$ is a translation by $k \in H^2(\pi; B/\Gamma_2B)$ of a homomorphism and that therefore the image of $\text{end}_\pi B/\Gamma_2B$ under c is fibrant. As $\text{haut}_\pi B/\Gamma_2B$ is a union of components of $\text{end}_\pi B/\Gamma_2B$, the same holds for the image of $\text{haut}_\pi B/\Gamma_2B$ under c and the desired result now readily follows.

§7. Automorphisms of diagrams of π-kernels

In this section we obtain a lemma (7.3) on automorphisms of diagrams of π-kernels in the sense of Eilenberg–Maclane, which will be used in § 8 to prove Theorem 6.1.

We start with a brief discussion of

7.1. π-KERNELS AND CENTRAL MAPS BETWEEN THEM. For a group G, let ζG denote its *center*, *aut* G its group of *automorphisms*, *in* $G \approx G/\zeta G$ its group of *inner automorphisms* and $\text{out}\, G = (\text{aut}\, G)/(\text{in}\, G)$ its group of *outer automorphisms*. Given a group π, a *π-kernel* then is [9, Ch. IV] a pair (G, ψ) where G is a group and $\psi: \pi \to \text{out}\, G$ a homomorphism. Similarly we define a *central map* $(G, \psi) \to (G', \psi')$ between two π-kernels as a pair (g, p) consisting of a homomorphism $g: G \to G'$ which sends ζG into $\zeta G'$ and a homomorphism

$$\pi \times_{\text{out}\, G} \text{aut}\, G \xrightarrow{p} \pi \times_{\text{out}\, G'} \text{aut}\, G'$$

over π which, over the identity of π, agrees with the homomorphism *in* $G \approx G/\zeta G \to G'/\zeta G' \approx$ *in* G' induced by g.

7.2. EXAMPLE. If $1 \to B \to C \to \pi \to 1$ is as in 4.1 (i) and (ii), then each $B_n/\Gamma_i B_n$ $(n \geq 0, i > 1)$ is a π-kernel in an obvious manner and all face operators between them become central maps [10, p. 347]. Moreover the same holds for the degeneracy operators if each $B_n/\Gamma_i B_n$ is nilpotent of class exactly $i-1$; otherwise they need not be "center preserving". This is automatic if C_0 and hence B_0 is free on more than one generator.

7.3. LEMMA. *Let π be a finite group, let D be a finite category* (i.e. its nerve is a finite simplicial set) *and let F be a functor from D to the category of π-kernels and* *central maps between them such that, for every object $d \in D$, Fd is finitely generated, nilpotent and torsion free as a group. Then the group* $\mathrm{aut}_\pi F$ *of self natural equivalences of F is of finite type.*

To prove this we will freely use some elementary algebraic group theory as can be found, for instance, in [6, §21] and [4, IV, 2.2].

Proof. First we consider the case that D has only one object and its identity map and show, essentially following Baumslag [1, Ch. 4]: if F is a π-kernel which is finitely generated, nilpotent and torsion free as a group, then $\mathrm{aut}_\pi F$ is of finite type.

We start with proving that the group $\mathrm{aut}\, F$ of *group automorphisms* of F is of finite type. Let MF denote the (uniquely divisible nilpotent) Malcev completion of F [1, p. 50] and let LF be the finite dimensional nilpotent Lie algebra over the field Q of the rationals associated with F [1, p. 48]. The Baker–Campbell–Hausdorff formula gives rise to a natural set isomorphism $\log: MF \approx LF$, the group F admits a natural embedding $F \subset MF$, there exists [1, p. 51] at least one *lattice subgroup* of MF containing F (i.e. a group $F \subset F' \subset MF$ such that $\log F' \subset LF$ is a free abelian group which spans LF as a vector space) and the intersection $\bar{F}$ of all such lattice subgroups is itself a lattice subgroup, which is natural in F and contains F as a subgroup of finite index [13]. Moreover, as F and $\bar{F}$ are nilpotent, $\mathrm{aut}\, F$ is of finite index in $\mathrm{aut}\, \bar{F}$ [1, p. 61]. Furthermore $\mathrm{aut}\, \bar{F}$ is isomorphic to the subgroup of $\mathrm{aut}\, LF$ consisting of those Lie algebra automorphisms that carry $\log \bar{F}$ isomorphically onto itself (called the *stabilizer* in $\mathrm{aut}\, LF$ of the lattice $\log \bar{F}$). By construction $\mathrm{aut}\, LF$ is the group of rational points of a linear algebraic group $\mathbf{aut}\, LF$ over Q (which operates on the vector space LF). By definition the stabilizer in $\mathrm{aut}\, LF$ of the lattice $\log \bar{F}$ is an arithmetic subgroup of $\mathrm{aut}\, LF$. As [2, §11] *every arithmetic subgroup of an algebraic group is of finite type*, so is $\mathrm{aut}\, \bar{F}$ and therefore $\mathrm{aut}\, F$.

Next we note that, using the Baker–Campbell–Hausdorff formula, it is not difficult to verify that $M(F/\zeta F)$ is in a natural manner the group of rational points of a unipotent algebraic group $\mathbf{M}(F/\zeta F)$ over Q. Conjugation produces a monomorphism $M(F/\zeta F) \to \mathrm{aut}\, LF$ which is easily seen to lift to an algebraic group map $\mathbf{M}(F/\zeta F) \to \mathbf{aut}\, LF$. The image of this map is contained in the unipotent radical of $\mathbf{aut}\, LF$ and is therefore a closed algebraic subgroup of $\mathbf{aut}\, LF$, denoted by $\mathbf{in}\, LF$. It follows that the image of $M(F/\zeta F)$ in $\mathrm{aut}\, LF$ (which we denote by $\mathrm{in}\, LF$) is the group of rational points of $\mathbf{in}\, LF$.

Finally lift the structure homomorphism $\psi: \pi \to \mathrm{out}\, F$ to a function $\varphi: \pi \to \mathrm{aut}\, F$ and note that, for every element $y \in \pi$, the function $\mathrm{com}\, \varphi y: \mathrm{aut}\, LF \to \mathrm{aut}\, LF$ given by $t \to [t, L\varphi y]$ gives rise to a subset $(\mathrm{com}\, \varphi y)^{-1}(\mathrm{in}\, LF) \subset \mathrm{aut}\, LF$

which is readily verified to be a subgroup and not to depend on the choice of φ. The functions com φy clearly lift to algebraic maps $\mathbf{com}\ \varphi y : \mathbf{aut}\ LF \to \mathbf{aut}\ LF$ and, using the fact that $\mathbf{in}\ LF$ is a closed algebraic subgroup of $\mathbf{aut}\ LF$, it is not difficult to see that each $(\mathbf{com}\ \varphi\psi)^{-1}(\mathbf{in}\ LF)$ is a closed algebraic subgroup of $\mathbf{aut}\ LF$. If

$$\mathrm{aut}_\pi\, LF = \bigcap_{y\in\pi} (\mathrm{com}\ \varphi y)^{-1}(\mathrm{in}\ LF) \quad \text{and} \quad \mathbf{aut}_\pi\, LF = \bigcap_{y\in\pi} (\mathbf{com}\ \varphi y)^{-1}(\mathbf{in}\ LF)$$

then it follows that $\mathbf{aut}_\pi\, LF$ is a closed algebraic subgroup of $\mathbf{aut}\ LF$ which has $\mathrm{aut}_\pi\, LF$ as its group of rational points. Moreover a straightforward calculation yields that an element of aut F is in $\mathrm{aut}_\pi\, F$ if and only if its image in aut LF is contained in $\mathrm{aut}_\pi\, LF$ and from this it readily follows that $\mathrm{aut}_\pi\, F$ is of finite type.

To prove Lemma 7.3 in general, let F now be a functor as in 7.3. Then LF is a functor from the category D to the category of Lie algebras and it is easy to check that its group of self natural equivalences aut LF is the group of rational points of a linear algebraic group $\mathbf{aut}\ LF$ which operates on the vector space $\oplus_{d\in D} LFd$, that aut F (the group of self natural group automorphisms) is isomorphic to a subgroup of finite index of the stabilizer in aut LF of the lattice $\oplus_{d\in D} \log \overline{Fd}$ and that the subgroup $\mathrm{aut}'_\pi\, LF \subset \mathrm{aut}\ LF$ given by the pull back diagram

$$\begin{array}{ccc} \mathrm{aut}'_\pi\, LF & \longrightarrow & \mathrm{aut}\ LF \\ \downarrow & & \downarrow \\ \prod_{d\in D} \mathrm{aut}_\pi\, LFd & \longrightarrow & \prod_{d\in D} \mathrm{aut}\ LFd \end{array}$$

is the group of rational points of a closed algebraic subgroup $\mathbf{aut}'_\pi\, LF \subset \mathbf{aut}\ LF$.

For every object $d \in D$, now identify $M(Fd/\zeta Fd)$ with in LFd under the obvious isomorphism, lift the structure homomorphism $\psi_d : \pi \to \mathrm{out}\ Fd$ to a function $\varphi_d : \pi \to \mathrm{aut}\ Fd$ and let $p_d : \mathrm{aut}'_\pi\, LF \to \mathrm{aut}_\pi\, LFd$ denote the projection, and, for every map $f : d \to d' \in D$, denote by f_* both the induced map $M(Fd/\zeta Fd) \to M(Fd'/\zeta Fd')$ and the structure map

$$\pi \times_{\mathrm{out}\,Fd} \mathrm{aut}\ Fd \to \pi \times_{\mathrm{out}\,Fd'} \mathrm{aut}\ Fd'$$

Then we denote by $\mathrm{aut}_\pi\, LF \subset \mathrm{aut}'_\pi\, LF$ the intersection of the equalizers of the diagrams

$$\mathrm{aut}'_\pi\, LF \underset{(\mathrm{com}\, f_*\varphi_d y)\cdot p_{d'}}{\overset{f_*\cdot(\mathrm{com}\,\varphi_d y)\cdot p_d}{\rightrightarrows}} M(Fd'/\zeta Fd')$$

where f runs through the maps of D and y runs through the elements of π. As the obvious maps $\mathbf{M}(Fd/\zeta Fd) \to \mathbf{in}\, LFd$ are algebraic maps of unipotent groups over Q which induce isomorphisms $M(Fd/\zeta Fd) \approx \text{in}\, LFd$ on the groups of rational points, these maps are isomorphisms themselves. Using this it is not difficult to verify that $\text{aut}_\pi LF$ is the group of rational points of a closed algebraic subgroup $\mathbf{aut}_\pi LF \subset \mathbf{aut}'_\pi LF$. Once again a straightforward calculation yields that an element of $\text{aut}\, F$ is in $\text{aut}_\pi F$ of and only if its image in $\text{aut}\, LF$ is contained in $\text{aut}_\pi LF$, readily implying the desired result.

§ 8. Final reduction

We now complete the proof of Theorem 1.1 by reducing Theorem 6.1 to Lemma 7.3.

First we note that it is not difficult to verify (by obstruction theory) *that the groups* $\pi_n\, \text{haut}_\pi C/\Gamma_i B$ $(n \geq 1, i \geq 1)$ *are finitely generated abelian and hence of finite type.* This fact, together with 2.9, 2.10 and 6.2 readily implies that if suffices to show that *the groups* $(\text{aut}_\pi C/\Gamma_i B)_n$ *are of finite type for all* $n \geq 0$.

To do this make the harmless assumption that C_0 is free on more than one generator. Then (7.2) $B/\Gamma_i B$ is a simplicial object over the category of π-kernels and central maps between them and one can define in the obvious manner its simplicial group of automorphisms $\text{aut}_\pi B/\Gamma_i B$. Now construct a sequence of maps.

$$1 \to Z^1(\pi; \Gamma_{i-1}B_n/\Gamma_i B_n) \xrightarrow{a} (\text{aut}_\pi C/\Gamma_i B)_n \xrightarrow{b} (\text{aut}_\pi B/\Gamma_i B)_n$$
$$\xrightarrow{c} H^2(\pi; \Gamma_{i-1}B_n/\Gamma_i B_n) \to 1$$

as follows (cf. § 6 and [9, Ch. IV]).

(i) The map b is the homomorphism induced by the functor which (see 7.2) assigns to every epimorphism $H \to \pi$, its kernel as a π-kernel and to every "center of the kernel preserving" map between such epimorphisms, the induced central map.

(ii) For every 1-cocycle $z \in Z^1(\pi, \Gamma_{i-1}B_n/\Gamma_i B_n)$, the map $az : \Delta[n] \otimes C/\Gamma_i B \to C/\Gamma_i B$ assigns to k-simplices $p \in \Delta[n]$ and $q \in C/\Gamma_i B$, the k-simplex $p'(zq') \cdot q \in C/\Gamma_i B$, where q' denotes the image of q in π and p' is the simplicial operator such that $p = p' i_n$, where i_n denotes the non-degenerate n-simplex of $\Delta[n]$. *Clearly a is a homomorphism.*

(iii) Restriction of a simplex $r : \Delta[n] \otimes B/\Gamma_i B \to B/\Gamma_i B \in (\text{aut}_\pi B/\Gamma_i B)_n$ to the simplex $i_n \in \Delta[n]$ yields an automorphism $r' : B_n/\Gamma_i B_n \to B_n/\Gamma_i B_n$. If $r'_* k$ denotes the extension of $B_n/\Gamma_i B_n$ by π induced by r' from the given extension

$k : B_n/\Gamma_i B_n \to C_n/\Gamma_i B_n \to \pi$, then $cr \in H^2(\pi; \Gamma_{i-1}B_n/\Gamma_i B_n)$ will be the element which (see [9, Ch. IV]) sends the equivalence class of $r'_* k$ to that of k. It is not difficult to verify that *the map c so defined is a crossed homomorphism.*

A simplified version of the argument that appears in the proof of 6.2 now yields that *the above sequence is exact* and that thus our problem is reduced to showing that *the groups* $(\text{aut}_\pi B/\Gamma_i B)_n$ *are of finite type for all* $n \geq 0$.

Let $c\Delta[n]$ denote the category of $\Delta[n]$ (i.e. $c\Delta[n]$ has as objects the simplices of $\Delta[n]$ and has a map $p \to q$ for every simplicial operator f such that $fp = q$) and let E be the functor from $c\Delta[n]$ to the category of π-kernels and central maps which, to each k-simplex of $\Delta[n]$, assigns $B_k/\Gamma_i B_k$. Then the group $\text{aut}_\pi E$ of self natural equivalences of E is clearly isomorphic to $(\text{aut}_\pi B/\Gamma_i B)_n$. Furthermore, let r be an integer such that all non-degenerate generators of C (and hence of B) are in dimensions $\leq r$, let $c^{n+r}\Delta[n] \subset C\Delta[n]$ denote the full subcategory generated by the simplices in dimensions $\leq n+r$ and let F be the restriction of E to $c^{n+r}\Delta[n]$. Then one readily verifies that the restriction $\text{aut}_\pi E \to \text{aut}_\pi F$ is an isomorphism and the desired result thus follows from 7.3.

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