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SIMPLICIAL LOCALIZATIONS OF CATEGORIES*

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1. Introduction

1.1. The simplicial localization. The *localization* of a category **C** with respect to a subcategory **W** is the category $\mathbf{C}[\mathbf{W}^{-1}]$ which has the same objects as **C** and is obtained from **C** by formally inverting the maps of **W**. Our purpose here is to show that $\mathbf{C}[\mathbf{W}^{-1}]$ reflects just one aspect of a much richer object, the *simplicial localization* $L\mathbf{C}$. This simplicial localization $L\mathbf{C}$ is a simplicial category with in each dimension the same objects as **C** (i.e. for every two objects $X, Y \in \mathbf{C}$, the maps $X \to Y \in L\mathbf{C}$ form a simplicial set $L\mathbf{C}(X, Y)$), which *has the localization* $\mathbf{C}[\mathbf{W}^{-1}]$ *as its "category of components"*. By this we mean that, for every two objects $X, Y \in \mathbf{C}$,

$$\pi_0 L\mathbf{C}(X, Y) \approx \mathbf{C}[\mathbf{W}^{-1}](X, Y)$$

i.e. the components of $L\mathbf{C}(X, Y)$ are (in 1–1 correspondence with) the maps $X \to Y \in \mathbf{C}[\mathbf{W}^{-1}]$.

In the present paper we define this simplicial localization and develop some of its basic theory. A more thorough homotopy theoretical analysis of the simplicial sets $L\mathbf{C}(X, Y)$ will be given in [4], partly for its own sake, and partly in preparation for [5], where we deal with our main

1.2. Application and justification. Let **C** be a *closed simplicial model category* in the sense of Quillen [10], i.e. **C** comes with three classes of maps (cofibrations, fibrations and weak equivalences) satisfying certain axioms and C has an additional simplicial structure which makes it possible to define, for every two objects $X, Y \in \mathbf{C}$, a *function complex* $\hom(X, Y)$. If one takes for $\mathbf{W} \subset \mathbf{C}$ the subcategory of the weak

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equivalences, then we show in [5] that, for every cofibrant object X and fibrant object Y, $L\mathbf{C}(X, Y)$ *has the same homotopy type as* $\hom(X, Y)$. Thus one can use the simplicial localization to define functorial function complexes in an arbitrary model category, even if there is no additional simplicial structure. Moreover these function complexes depend only on the weak equivalences and not on the cofibrations or the fibrations.

1.3. Organization of the paper. After fixing some notation and terminology (in 1.4) we start, in Sections 2 and 3, with some results on *free categories* and their *localizations* and observe that *free categories are better behaved under localizations* than more general categories. Guided by this we define, in Section 4, the (*standard*) *simplicial localization* $L(\mathbf{C}, \mathbf{W})$ (for short $L\mathbf{C}$) as the *dimensionwise localization of the standard free simplicial resolution of* $\mathbf{C}$. We also list there some immediate consequences of this definition and observe that any other natural free simplicial resolution would have done as well, as least up to homotopy. In Section 5 we note the not so obvious fact that, *for subcategories* $\mathbf{W}_1, \mathbf{W}_2 \subset \mathbf{C}$, *the simplicial localizations* $L(\mathbf{C}, \mathbf{W}_1)$ *and* $L(\mathbf{C}, \mathbf{W}_2)$ *are homotopically equivalent whenever* $\mathbf{C}[\mathbf{W}_1^{-1}] = \mathbf{C}[\mathbf{W}_2^{-1}]$. We also discuss there a few extreme cases of simplicial localizations. In Section 6 we consider the obvious generalization of the definition of simplicial localization from pairs of categories to pairs of "simplicial categories with (see 1.4) the same objects in each dimension."

Except for two lemmas (6.2 and 6.4) our results up to this point are not hard to prove, once they have been formulated. To prove these two lemmas, as well as to put things a bit more in perspective, in the remaining sections of the paper we approach the simplicial localization from the point of view of *homotopical algebra*. In Sections 7 and 8 we note that *the category* sO-**Cat** (see 1.4) of simplicial categories with a fixed set O of objects, *can be given the structure of a closed simplicial model category*. We also discuss there in some detail the notions of cofibration and pushout in this category and prove that sO-**Cat** is a proper model category, i.e. the pushout (resp. pullback) of a weak equivalence along a cofibration (resp. fibration) is again a weak equivalence. Finally, in Section 9, we deal with the *groupoid completion* in sO-**Cat**, which can be considered as an absolute version of the localization and, in Section 10, with the *localization* itself and, of course, a proof of Lemmas 6.2 and 6.4.

1.4. Notation, terminology, etc. (i) *Categories.* Except for the categories O-**Cat** and sO-**Cat** defined below, *all categories will be small* and, unless otherwise noted, *subcategories will contain all the objects.* If $\mathbf{C}$ is a category and $X, Y \in \mathbf{C}$ are objects, then $\mathbf{C}(X, Y)$ will denote the set of maps $X \to Y \in \mathbf{C}$.

(ii) *The category* O-**Cat**. Let O be an arbitrary but *fixed* set (of objects). Then O-**Cat** will be the category with as objects *the small categories which have O as their set of objects* and with as maps *the functors which are the identity on the objects.* For instance, if O consists of only one element, then O-**Cat** is just the category of monoids.

(iii) *Free products in O*-**Cat**. The categorical sum in *O*-**Cat** will be denoted by $*$ and called *free product*. Theire is an obvious 1–1 correspondence between the *non-identity maps of a free product* $\mathbf{C} * \mathbf{D}$ *and the finite compositions of non-identity maps of* $\mathbf{C}$ *and* $\mathbf{D}$ *in which no two adjacent maps are both in* $\mathbf{C}$ *or both in* $\mathbf{D}$.

(iv) *The category* s*O*-**Cat**. This is the category of simplicial objects over *O*-**Cat**. An object $\mathbf{B} \in$ s*O*-**Cat** thus is a simplicial category with the same set O of objects in each dimension. The category *O*-**Cat** is sometimes identified with the subcategory of s*O*-**Cat** of the "discrete" simplicial objects.

(v) *Weak equivalences in* s*O*-**Cat**. These are the maps $\mathbf{A} \to \mathbf{B} \in$ s*O*-**Cat** which, for every two objects $X, Y \in O$, induce a weak homotopy equivalence of simplicial sets $\mathbf{A}(X, Y) \sim \mathbf{B}(X, Y)$.

(vi) *The nerve functors for O*-**Cat** *and* s*O*-**Cat**. For a category $\mathbf{C} \in O$-**Cat**, its nerve is, as usual [11], the simplicial set $N\mathbf{C}$ which has as its k-simplices the sequences

$$X_k \to X_{k-1} \to \cdots \to X_0$$

of maps in $\mathbf{C}$. Dimensionwise application of this functor N to an object $\mathbf{B} \in$ s*O*-**Cat** thus yields a bisimplicial set and the *diagonal* of the latter will be denoted by $N\mathbf{B}$ and called the *nerve* of $\mathbf{B}$.

(vii) *Homotopy invariance of the diagonal of a bisimplicial set.* We will often use, explicitly or implicitly, the following result [3, Chapter XII, Section 4].

If $K \to L$ *is a map of bisimplicial sets such that, for every integer* $i \geq 0$, *the restriction* $K_{i,*} \to L_{i,*}$ *is a weak homotopy equivalence, then its diagonal* $\operatorname{diag} K \to \operatorname{diag} L$ *is also a weak homotopy equivalence.*

This readily implies:

If $\mathbf{A} \to \mathbf{B} \in$ s*O*-**Cat** *is a weak equivalence, then the induced map* $N\mathbf{A} \to N\mathbf{B}$ *is a weak homotopy equivalence.*

(viii) *Main references.* The paper is essentially self contained, except for the last four sections which rely heavily on the results of Quillen [10] and once (in Section 8) on May [9]. Of course we assume a knowledge of basic simplicial homotopy theory such as can be found for instance in [8] or [3, Chapter VIII].

2. Free categories

We start with a few results on free categories which will be needed later. First we recall the definition [7, p. 50].

2.1. Free categories. A category $\mathbf{C} \in O$-**Cat** (1.4) is called *free* if there exists a set S of non-identity maps in $\mathbf{C}$ such that every non-identity map in $\mathbf{C}$ can uniquely be written as a finite composition of maps in S. If such a set S exists, it is clearly unique; its elements are called the *generators of* $\mathbf{C}$.

One readily verifies (see 1.4)

2.2. Proposition. *A free product of free categories is a free category.*

2.3. Proposition. *Every free category is the free product of free categories with only one generator.*

An important example of a free category is provided by

2.4. The free category FC on a category C. Let $\mathbf{C} \in O$-**Cat** (1.4). Then the *free category* on **C** is the free category $F\mathbf{C} \in O$-**Cat** which has a generator Fc for every non-identity map $c \in \mathbf{C}$. Of course, as with any such free construction, there are functors

$$F\mathbf{C} \xrightarrow{\varphi} \mathbf{C} \quad \text{and} \quad F\mathbf{C} \xrightarrow{\psi} F^2\mathbf{C}$$

given by $Fc \mapsto c$ and $Fc \mapsto F(Fc)$ respectively, which satisfy the *comonad* (*cotriple*) *identities* [7, p. 135]

$$\varphi(F\varphi) = \varphi(\varphi F), \qquad (F\psi)\psi = (\psi F)\psi, \qquad (F\varphi)\psi = \mathrm{id} = (\varphi F)\psi.$$

They therefore can be used to construct

2.5. The standard resolution $F_*\mathbf{C}$. Let $\mathbf{C} \in O$-**Cat** be a category. Then the *standard resolution of* **C** is the *simplicial category* $F_*\mathbf{C} \in \mathrm{s}O$-**Cat** (1.4) which in dimension k consists of the category $F_k\mathbf{C} = F^{k+1}\mathbf{C}$, and which has the functors

$$(F_k\mathbf{C} \xrightarrow{d_i} F_{k-1}\mathbf{C}) = (F^{k+1}\mathbf{C} \xrightarrow{F^i\varphi F^{k-i}} F^k\mathbf{C}),$$

$$(F_k\mathbf{C} \xrightarrow{s_i} F_{k+1}\mathbf{C}) = (F^{k+1}\mathbf{C} \xrightarrow{F^i\psi F^{k-i}} F^{k+2}\mathbf{C})$$

as its face and degeneracy functors.

There is, of course, also the map (i.e. functor) $\varphi : F_*\mathbf{C} \to \mathbf{C} \in \mathrm{s}O$-**Cat** (1.4) given by

$$(F_k\mathbf{C} \xrightarrow{\varphi} \mathbf{C}) = (F^{k+1}\mathbf{C} \xrightarrow{\varphi^{k+1}} \mathbf{C})$$

which has the useful property

2.6. Proposition. *The map $\varphi : F_*\mathbf{C} \to \mathbf{C} \in \mathrm{s}O$-**Cat** is a weak equivalence* (1.4).

Proof. For every two objects $X, Y \in \mathbf{C}$, the function $a \mapsto Fa$ yields a contracting homotopy of $F_*\mathbf{C}(X, Y)$ onto $\mathbf{C}(X, Y)$.

For later reference we mention the somewhat related

2.7. Proposition. *If two maps $\mathbf{A} \to \mathbf{A}'$ and $\mathbf{B} \to \mathbf{B}' \in \mathrm{s}O$-**Cat** are weak equivalences, then so is their free product $\mathbf{A} * \mathbf{B} \to \mathbf{A}' * \mathbf{B}'$* (1.4).

Proof. It clearly suffices to prove the case $\mathbf{B}=\mathbf{B}'$, and this is done by

(i) first proving the case that $\mathbf{B}$ *is discrete and free* (which is straightforward),

(ii) then, using a diagonal argument (1.4 (vii)), extending it to the case that $\mathbf{B}$ *is* not necessarily discrete, but still *free in each dimension*, and

(iii) finally observing that the argument of the proof of 2.6 also yields that, for all $\mathbf{B}$, the obvious maps $\mathbf{A} * F_* \mathbf{B} \to \mathbf{A} * \mathbf{B}$ and $\mathbf{A}' * F_* \mathbf{B} \to \mathbf{A}' * \mathbf{B}$ are weak equivalences.

We end this section with a brief discussion of the homotopy type of

2.8. The nerve of a free category. For a free category $\mathbf{C} \in O$-**Cat** one can consider not only its nerve $N\mathbf{C}$ (1.4), but also, for every integer $k \geq 1$, the k-dimensional subcomplex $N^k \mathbf{C} \subset N\mathbf{C}$ generated by the k-simplices

$$X_k \to X_{k-1} \to \cdots \to X_0$$

of $N\mathbf{C}$, for which each of the maps $X_i \to X_{i-1}$ is either a generator of $\mathbf{C}$ or an identity map. One then has

2.9. Proposition. *Let* $\mathbf{C} \in O$-**Cat** *be a free category. Then, for every integer* $k \geq 1$, *the inclusion* $N^k \mathbf{C} \subset N\mathbf{C}$ *is a weak homotopy equivalence.*

Proof. It is not hard to verify that, for every integer $k \geq 1$, the geometric realization $|N^k \mathbf{C}|$ is a strong deformation retract of $|N^{k+1}\mathbf{C}|$. The desired result now follows from the fact that $N\mathbf{C}$ is the union of the $N^k \mathbf{C}$.

3. The (old) localization

Next we briefly review the (old) notion of localization and note (3.7) that, in the *free* case, this localization does not affect the homotopy type of the nerve.

3.1. The localization of a category with respect to a subcategory. Let $\mathbf{C} \in O$-**Cat** be a category and $\mathbf{W} \subset \mathbf{C}$ a subcategory (1.4). The $\mathbf{W}$-*localization of* $\mathbf{C}$ then is [6] the category $\mathbf{C}[\mathbf{W}^{-1}]$ obtained from $\mathbf{C}$ by formally inverting all maps of $\mathbf{W}$. In other words, $\mathbf{C}[\mathbf{W}^{-1}]$ has the same objects as $\mathbf{C}$ and its maps are obtaned from the "composable words" in the maps of $\mathbf{C}$ and the formal inverses of the maps of $\mathbf{W}$, by means of the obvious equivalence relation [6].

3.2. Example. If $\mathbf{C} = \mathbf{D} * \mathbf{W}$ (1.4), then

$$\mathbf{C}[\mathbf{W}^{-1}] = \mathbf{D} * \mathbf{W}[\mathbf{W}^{-1}] \quad \text{and} \quad \mathbf{C}[\mathbf{C}^{-1}] = \mathbf{D}[\mathbf{D}^{-1}] * \mathbf{W}[\mathbf{W}^{-1}].$$

If moreover $\mathbf{D}$ and $\mathbf{W}$ (and hence $\mathbf{C}$) are free categories then the non-identity maps of $\mathbf{C}[\mathbf{W}^{-1}]$ are (in 1–1 correspondence with) the non-empty "composable reduced words" in the generators of $\mathbf{C}$ and the formal inverses of the generators of $\mathbf{W}$.

3.3. The universal functor $p:\mathbf{C}\to\mathbf{C}[\mathbf{W}^{-1}]$. Let $\mathbf{C}\in O$-**Cat** be a category and $\mathbf{W}\subset\mathbf{C}$ a subcategory. Then the obvious functor $p:\mathbf{C}\to\mathbf{C}[\mathbf{W}^{-1}]$ is *initial* among the functors from $\mathbf{C}$ which send all maps of $\mathbf{W}$ into equivalences.

Related to this is the notion of

3.4. Closed subcategories. Let $\mathbf{C}\in O$-**Cat** be a category. A subcategory $\mathbf{W}\subset\mathbf{C}$ then will be called *closed* (in $\mathbf{C}$) if $\mathbf{W}$ consists exactly of those maps of $\mathbf{C}$ which go to equivalences under the functor $p:\mathbf{C}\to\mathbf{C}[\mathbf{W}^{-1}]$. Similarly the *closure* of a subcategory $\mathbf{W}\subset\mathbf{C}$ will be the smallest closed subcategory of $\mathbf{C}$ containing $\mathbf{W}$.

These definitions immediately imply

3.5. Proposition. *Let* $\mathbf{C}\in O$-**Cat** *be a category,* $\mathbf{W}\subset\mathbf{C}$ *a subcategory and* $\bar{\mathbf{W}}$ *its closure. Then the induced map* $\mathbf{C}[\mathbf{W}^{-1}]\to\mathbf{C}[\bar{\mathbf{W}}^{-1}]$ *is an equivalence.*

3.6. Corollary. *Let* $\mathbf{C}\in O$-**Cat** *be a category. Then the (equivalence classes of) localizations of* $\mathbf{C}$ *are in* 1–1 *correspondence with the closed subcategories of* $\mathbf{C}$.

We end with observing that, although (see [1]) the functor $p:\mathbf{C}\to\mathbf{C}[\mathbf{W}^{-1}]$ in general does *not* preserve the homotopy type of the nerve, one has

3.7. Proposition. *If* $\mathbf{C}=\mathbf{D}*\mathbf{W}$, *where* $\mathbf{W}$ *is a free category, then the map*

$$N(\mathbf{D}*\mathbf{W})=N\mathbf{C}\xrightarrow{Np}N\mathbf{C}[\mathbf{W}^{-1}]=N[\mathbf{W}^{-1}]=N(\mathbf{D}*\mathbf{W}[\mathbf{W}^{-1}])$$

is a weak homotopy equivalence.

Proof. This is easy to verify when $\mathbf{C}$ is a free category on one generator. The general case then follows using 2.3, 3.2 and the following

3.8. Lemma. *Let* $\mathbf{C}=\mathbf{D}*\mathbf{E}$. *Then the inclusion*

$$N\mathbf{D}\cup N\mathbf{E}\to N(\mathbf{D}*\mathbf{E})=N\mathbf{C}$$

is a weak homotopy equivalence.

Proof. If $\mathbf{D}$ and $\mathbf{E}$ are free, then this follows from 2.9. To prove the general case, one considers the commutative diagram (1.4 and 2.5)

$$\begin{tikzcd} NF_*\mathbf{D}\cup NF_*\mathbf{E} \arrow[r] \arrow[d] & N(F_*\mathbf{D}*F_*\mathbf{E}) \arrow[d] \\ N\mathbf{D}\cup N\mathbf{E} \arrow[r] & N(\mathbf{D}*\mathbf{E})=N\mathbf{C} \end{tikzcd}$$

and notes that

(i) The top map is a weak homotopy equivalence in view of 1.4 (vii) and the fact that the result has already been proved for the free case, and

(ii) the vertical maps are also weak homotopy equivalences, in view of 1.4 (vii), 2.6 and 2.7.

4. The (standard) simplicial localization

Now we finally define the simplicial localization in a manner which is suggested by Propositions 2.6 and 3.7. We also list a few of the most immediate consequences of this definition and mention some (of course homotopically equivalent) variations.

4.1. The (standard) simplicial localization. Let $\mathbf{C} \in O$-**Cat** be a category and $\mathbf{W} \subset \mathbf{C}$ a subcategory. Then the *(standard) simplicial localization* of $\mathbf{C}$ with respect to $\mathbf{W}$ is the simplicial category $L(\mathbf{C}, \mathbf{W}) \in sO$-**Cat** defined by (see 2.5 and 3.1)

$$L(\mathbf{C}, \mathbf{W}) = F_*\mathbf{C}[F_*\mathbf{W}^{-1}].$$

When no confusion can arise, we will *abbreviate* $L(\mathbf{C}, \mathbf{W})$ to $L\mathbf{C}$.

This definition immediately implies

4.2. The simplicial localization has the (old) localization as its "category of components". *For every two objects $X, Y \in \mathbf{C}$, the components of $L\mathbf{C}(X, Y)$ are in 1–1 correspondence with the maps $X \to Y \in \mathbf{C}[\mathbf{W}^{-1}]$, i.e.*

$$\pi_0 L\mathbf{C} = \mathbf{C}[\mathbf{W}^{-1}].$$

Combining 1.4 (vii) with propositions 2.6 and 3.7 one gets

4.3. The simplicial localization preserves the homotopy type of the nerve. By this we mean that *the obvious maps*

$$N\mathbf{C} \leftarrow NF_*\mathbf{C} \to N(F_*\mathbf{C}[F_*\mathbf{W}^{-1}]) = NL\mathbf{C}$$

are weak homotopy equivalences.

Also easy to prove is

4.4. Proposition. *A map $u : X \to Y \in \mathbf{C}$ is in $\mathbf{W}$ if and only if, for every object $V \in \mathbf{C}$, "composition with u" yields isomorphisms*

$$L\mathbf{C}(V, X) \overset{u_*}{\approx} L\mathbf{C}(V, Y) \quad \text{and} \quad L\mathbf{C}(Y, V) \overset{u^*}{\approx} L\mathbf{C}(X, Y).$$

Note that these maps are not isomorphisms if u is *not* in $\mathbf{W}$, even if u is an equivalence. To prevent this from happening, one can assume that $\mathbf{W}$ is a *closed* subcategory (3.4) of $\mathbf{C}$, which, in view of 5.1, is no serious restriction.

We end with observing that there are, of course

4.5. Variations on the definition of simplicial localization. *Let* $\mathbf{C} \in O$-**Cat** *be a category and* $\mathbf{W} \subset \mathbf{C}$ *a subcategory. Moreover let* $\mathbf{B} \in \mathrm{s}O$-**Cat** *be free* (i.e. $\mathbf{B}$ is a free category in each dimension and all degeneracies of generators are generators), *let* $\mathbf{V} \subset \mathbf{B}$ *be a subcategory which is a free factor in each dimension and let* $S: \mathbf{B} \to \mathbf{C} \in \mathrm{s}O$-**Cat** *be a functor which sends all of* $\mathbf{V}$ *into* $\mathbf{W}$. *If* $S: \mathbf{B} \to \mathbf{C}$ *and its restriction* $S: \mathbf{V} \to \mathbf{W}$ *are both weak equivalences, then so are the obvious functors*

$$\mathbf{B}[\mathbf{V}^{-1}] \leftarrow \operatorname{diag} F_*\mathbf{B}[F_*\mathbf{V}^{-1}] \to F_*\mathbf{C}[F_*\mathbf{W}^{-1}] = L\mathbf{C}.$$

Proof. This follows immediately from homotopy Lemma 6.2, which in turn follows from Proposition 10.3.

4.6. Example. Take, for instance, $B = \operatorname{diag} F_*F_*\mathbf{C}$ and $\mathbf{V} = \operatorname{diag} F_*F_*\mathbf{W}$.

5. Some special cases

We note (5.2) that, *up to homotopy, the simplicial localizations of a category* $\mathbf{C}$ *are in* 1–1 *correspondence with the closed* (3.4) *subcategories of* $\mathbf{C}$, and show (5.3–5) that, in some extreme cases, the simplicial localization admits a simple description in terms of $\mathbf{C}$ and $\mathbf{W}$. First the

5.1. Closure lemma. *Let* $\mathbf{C} \in O$-**Cat** *be a category,* $\mathbf{W} \subset \mathbf{C}$ *a subcategory and* $\bar{\mathbf{W}}$ *its closure* (3.4). *Then the induced functor* $L(\mathbf{C}, \mathbf{W}) \to L(\mathbf{C}, \bar{\mathbf{W}}) \in \mathrm{s}O$-**Cat** *is a weak equivalence.*

Proof. This is a special case of closure Lemma 6.4 which in turn is a consequence of Proposition 10.5.

As in 3.6 the closure lemma implies

5.2. Corollary. *Let* $\mathbf{C} \in O$-**Cat** *be a category. Then the "weak equivalence classes" of simplicial localizations of* $\mathbf{C}$ *are in* 1–1 *correspondence with the closed subcategories of* $\mathbf{C}$.

In the following situations the simplicial localization does not really yield anything new.

5.3. The trivial case. *If* $\mathbf{W}$ *contains only isomorphisms, then the functor* $\pi_0: L\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ *is a weak equivalence.*

Proof. If $\mathbf{W}$ contains only identity maps, then $L\mathbf{C} = F_*\mathbf{C}$ and 5.3 reduces to 2.6. The general case now follows from 5.1.

5.4. The free case. *If* $\mathbf{C}=\mathbf{D}*\mathbf{W}$, *where* $\mathbf{W}$ *is a free category, then the functor* $\pi_0: L\mathbf{C} \to \mathbf{C}[\mathbf{W}^{-1}]$ *is a weak equivalence.*

Proof. This is not hard to prove using 2.6, 2.7, 3.2 and 6.2.

More interesting is

5.5. The "invert everything" case. *If* $\mathbf{W}=\mathbf{C}$ *and* $N\mathbf{C}$ *is connected, then*

(i) $L\mathbf{C}$ *is a simplicial groupoid and hence the simplicial sets* $L\mathbf{C}(X, Y)$ *are all isomorphic, while the simplicial sets* $L\mathbf{C}(X, X)$ *are actually simplicial groups and are all isomorphic as such, and*

(ii) *the classifying complex of* $L\mathbf{C}(X, X)$ *has the homotopy type of* $N\mathbf{C}$ *and each* $L\mathbf{C}(X, Y)$ *thus has the homotopy type of the loops on* $N\mathbf{C}$.

Proof. Part (i) is trivial. To prove part (ii) note that, in view of 2.9 and 4.3, for every integer $k \geq 0$, the universal covering map

$$\hat{N}(F_k\mathbf{C}[F_k\mathbf{C}^{-1}]) \to N(F_k\mathbf{C}[F_k\mathbf{C}^{-1}])$$

is a principal fibration with a contractible total complex and with $L\mathbf{C}(X, X)_k$ as fibre over X. Together these maps form a bisimplicial map and the desired result now follows by taking the diagonal and using 4.3.

6. A generalization

As the simplicial localization goes from (pairs of) categories to simplicial categories, it, not unexpectedly, admits the following

6.1. Generalization. Let $\mathbf{B} \in \mathrm{s}O\text{-}\mathbf{Cat}$ and let $\mathbf{V} \subset \mathbf{B}$ be a subcategory. The *simplicial localization* of $\mathbf{B}$ with respect to $\mathbf{V}$ then is the *simplicial category* $L(\mathbf{B}, \mathbf{V}) \in \mathrm{s}O\text{-}\mathbf{Cat}$ defined by

$$L(\mathbf{B}, \mathbf{V}) = \operatorname{diag} F_*\mathbf{B}[F_*\mathbf{V}^{-1}].$$

Again, when no confusion can arise, we will *abbreviate* $L(\mathbf{B}, \mathbf{V})$ *to* $L\mathbf{B}$.

Clearly the category of components is the (*old*) *localization*

$$\pi_0 L\mathbf{B} = (\pi_0\mathbf{B})[(\operatorname{im} \pi_0\mathbf{V})^{-1}]$$

and 4.3 readily implies that *this simplicial localization also preserves the homotopy type of the nerve*, in the sense that *the obvious maps*

$$N\mathbf{B} \leftarrow NF_*\mathbf{B} \to N(F_*\mathbf{B}[F_*\mathbf{V}^{-1}]) = NL\mathbf{B}$$

are weak homotopy equivalences.

Another useful but not surprising property is formulated in

6.2. Homotopy lemma. *Let* $\mathbf{A}$ *and* $\mathbf{B} \in s\mathcal{O}$-**Cat** *be free* (see (4.5)), *let* $\mathbf{U} \subset \mathbf{A}$ *and* $\mathbf{V} \subset \mathbf{B}$ *be subcategories which are a free factor in each dimension, and let* $S : \mathbf{A} \to \mathbf{B} \in s\mathcal{O}$-**Cat** *be a functor which sends all of* $\mathbf{U}$ *into* $\mathbf{V}$. *If* $S : \mathbf{A} \to \mathbf{B}$ *and its restriction* $S : \mathbf{U} \to \mathbf{V}$ *are both weak equivalences, then so is the induced map* $\mathbf{A}[\mathbf{U}^{-1}] \to \mathbf{B}[\mathbf{V}^{-1}]$.

6.3. Corollary. *Let* $\mathbf{A}, \mathbf{B} \in s\mathcal{O}$-**Cat**, *let* $\mathbf{U} \subset \mathbf{A}$ *and* $\mathbf{V} \subset \mathbf{B}$ *be subcategories and let* $S : \mathbf{A} \to \mathbf{B} \in s\mathcal{O}$-**Cat** *be a functor which sends all of* $\mathbf{U}$ *into* $\mathbf{V}$. *If* $S : \mathbf{A} \to \mathbf{B}$ *and its restriction* $S : \mathbf{U} \to \mathbf{V}$ *are both weak equivalences, then so is the induced map* $L\mathbf{A} \to L\mathbf{B}$.

However less expected is

6.4. Closure lemma. *Let* $\mathbf{B} \in s\mathcal{O}$-**Cat**, *let* $\mathbf{V} \subset \mathbf{B}$ *be a subcategory and let* $\bar{\mathbf{V}} \subset \mathbf{B}$ *be the* "*π_0-closure*" *of* $\mathbf{V}$ *in* $\mathbf{B}$, i.e. the counter image of $\overline{\pi_0\mathbf{V}} \subset \pi_0\mathbf{B}$, where $\overline{\pi_0\mathbf{V}}$ denotes the closure (3.4) in $\pi_0\mathbf{B}$ of the image of $\pi_0\mathbf{V}$. *Then the inclusion* $L(\mathbf{B}, \mathbf{V}) \to L(\mathbf{B}, \bar{\mathbf{V}})$ *is a weak equivalence.*

6.5. Corollary. *The* "*weak equivalence classes*" *of simplicial localizations of a simplicial category* $\mathbf{B} \in s\mathcal{O}$-**Cat** *are in* 1–1 *correspondence with the closed* (3.4) *subcategories of* $\pi_0\mathbf{B}$.

The proofs of Lemmas 6.2 and 6.4 are non-trivial and will only be given in Section 10, after the development of some more machinery in Sections 7–9.

7. A homotopical algebra approach

In the remaining sections of this paper we develop a homotopical algebra approach to the simplicial localization, which will put the results of the preceding sections more in perspective and simultaneously provide a proof for the not yet proven lemmas 6.2 and 6.4.

We start with the construction of

7.1. A closed simplicial model category structure for $s\mathcal{O}$-**Cat.** Consider the following three classes of maps in $s\mathcal{O}$-**Cat**:

(i) *Weak equivalences* (see 1.4).

(ii) *Fibrations.* These are the maps $\mathbf{A} \to \mathbf{B} \in s\mathcal{O}$-**Cat** which, for every pair of objects $X, Y \in \mathcal{O}$, induce a fibration of simplicial sets $\mathbf{A}(X, Y) \to \mathbf{B}(X, Y)$.

(iii) *Cofibrations.* These are the maps which have the left lifting property [10, I, p. 5.1] with respect to those fibrations which are weak equivalences.

One then has [10, I, Section 4]:

7.2. Proposition. *The category* $s\mathcal{O}$-**Cat**, *with the above classes of weak equivalences, fibrations and cofibrations and with the obvious simplicial structure, is a closed simplicial model category.*

In fact one even has

7.3. Proposition. *The above closed simplicial model category structure on* sO-**Cat** *is proper*, i.e. [2] whenever a square

$$\begin{tikzcd} \mathbf{C} \arrow[r, "i"] \arrow[d, "g"] & \mathbf{C}' \arrow[d, "g'"] \\ \mathbf{D} \arrow[r, "j"] & \mathbf{D}' \end{tikzcd}$$

in sO-**Cat** is

(i) a pullback with g' a fibration and j a weak equivalence, then i is a weak equivalence, and

(ii) a pushout with g a cofibration and i a weak equivalence, then j is a weak equivalence.

Proof. Part (i) is trivial (because *simplicial sets form a proper closed simplicial model category*), but part (ii) is not and its proof will therefore be postponed until the end of Section 8, after a more detailed investigation of pushouts.

We end with a more useful description of cofibrations in sO-**Cat** than the above one. To formulate it we need the following two definitions.

7.4. Free maps in sO-Cat. A map $f: \mathbf{A} \to \mathbf{B} \in$ sO-**Cat** will be called *free* if

(i) f is 1–1,

(ii) in each dimension k, $\mathbf{B}_k$ admits a (unique) free factorization $\mathbf{B}_k = f(\mathbf{A}_k) * \mathbf{F}_k$, in which $\mathbf{F}_k$ is a free category, and

(iii) for each $k \geq 0$, all degeneracies of generators of $\mathbf{F}_k$ are generators of $\mathbf{F}_{k+1}$.

In particular, *if* $\mathbf{A}$ *is the initial object of* sO-**Cat** (i.e. $\mathbf{A}$ has only identity maps), *then f is a free map if and only if* $\mathbf{B}$ *is free in the sense of* 4.5.

7.5. Strong retracts of maps. A map $f: \mathbf{A} \to \mathbf{B}$ is called a *strong retract* of a map $f': \mathbf{A} \to \mathbf{B}'$ if there exists a commutative diagram

$$\begin{tikzcd} & \mathbf{A} \arrow[ddl, "f"'] \arrow[d, "f'"] \arrow[ddr, "f"] & \\ & \mathbf{B}' \arrow[dr] & \\ \mathbf{B} \arrow[ur] \arrow[rr, "id"'] & & \mathbf{B} \end{tikzcd}$$

The argument of [10, II, p. 4.11] then yields the following

7.6. Characterization of the cofibrations in sO-Cat. *A map in* sO-**Cat** *is a cofibration if and only if it is a strong retract of a free map. In particular, an object in* sO-**Cat** *is cofibrant if and only if it is a retract of a free one.*

8. Pushouts in sO-Cat

Our main object in this section is to show (8.1) that *pushing out a cofibration has homotopy meaning in* sO-**Cat**, and then use this to prove the second half of Proposition 7.3.

8.1. Proposition. *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "f"'] & \mathbf{C} \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{D} \end{tikzcd} \longrightarrow \begin{tikzcd} \mathbf{A}' \arrow[r] \arrow[d, "f'"'] & \mathbf{C}' \arrow[d] \\ \mathbf{B}' \arrow[r] & \mathbf{D}' \end{tikzcd}$$

be a map between two pushout squares in sO-**Cat**, *which induces weak equivalences* $\mathbf{A} \sim \mathbf{A}'$, $\mathbf{B} \sim \mathbf{B}'$ *and* $\mathbf{C} \sim \mathbf{C}'$ *and assume that f and f' are cofibrations. Then the induced map* $\mathbf{D} \to \mathbf{D}'$ *is also a weak equivalence.*

Proof. This follows immediately from 2.7 and the following proposition.

8.2. Proposition. *Let*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "f"'] & \mathbf{C} \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{D} \end{tikzcd}$$

be a pushout square in sO-**Cat** *in which f is a cofibration and let* $(\mathbf{B}, \mathbf{A}, \mathbf{C})$ *denote the simplicial object over* sO-**Cat** (i.e. bisimplicial object over O-**Cat**) *which, in dimension* i, *consists of the free product with* $(i+2)$ *factors*

$$(\mathbf{B}, \mathbf{A}, \mathbf{C})_i = \mathbf{B} * \mathbf{A} * \cdots * \mathbf{A} * \mathbf{C}$$

and which has the obvious face and degeneracy operators [9, p. 69]. *Then the induced map*

$$\operatorname{diag}(\mathbf{B}, \mathbf{A}, \mathbf{C}) \to \mathbf{D} \in \mathrm{s}O\text{-}\mathbf{Cat}$$

is a weak equivalence.

Proof. In view of 7.6 we may assume that *f is a free map.* If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are discrete (i.e. in O-**Cat**), then there is a free category $\mathbf{F} \in O$-**Cat** such that $\mathbf{B} \approx \mathbf{F} * \mathbf{A}$ and $\mathbf{D} \approx \mathbf{F} * \mathbf{C}$. Hence

$$(\mathbf{B}, \mathbf{A}, \mathbf{C})_i \approx \mathbf{F} * \mathbf{A} * \cdots * \mathbf{A} * \mathbf{C}$$

is a free product with $(i+3)$ factors and it is not hard to construct a contracting homotopy which contracts $(\mathbf{B}, \mathbf{A}, \mathbf{C})$ onto $\mathbf{D}$. The non-discrete case follows by a diagonal argument.

Now we are ready for a

8.3. Proof of the second part of 7.3. In view of 7.6 we may assume that *g is a free map.* Next one observes that *each free map is the direct limit of a sequence of free maps in which all the non-degenerate generators are in the same dimension.* Thus one only has to show that, given a free category $\mathbf{F} \in O$-**Cat** and a diagram in sO-**Cat** [10, Chapter II]

$$\begin{tikzcd}
\mathbf{F} \otimes \dot{\Delta}[k] \arrow[r] \arrow[d, "\text{incl.}"] & \mathbf{C} \arrow[r, "i"] \arrow[d, "g"] & \mathbf{C}' \arrow[d, "g'"] \\
\mathbf{F} \otimes \Delta[k] \arrow[r] & \mathbf{D} \arrow[r, "j"] & \mathbf{D}'
\end{tikzcd}$$

in which all squares are pushouts and i is a weak equivalence, then the map j is also a weak equivalence. But this now follows immediately from Proposition 8.1.

9. Groupoid completions in sO-Cat

Localization is a *relative* notion; it is defined for maps of sO-**Cat**. We therefore first discuss here the corresponding *absolute* notion, which is also called

9.1. Groupoid completion. The *groupoid completion* of a simplicial category $\mathbf{V} \in$ sO-**Cat** is the dimensionwise $\mathbf{V}$-localization of $\mathbf{V}$, i.e. the simplicial category $\mathbf{V}[\mathbf{V}^{-1}] \in$ sO-**Cat** obtained from $\mathbf{V}$ by formally inverting all maps. This definition readily implies that

$$\pi_0(\mathbf{V}[\mathbf{V}^{-1}]) = (\pi_0\mathbf{V})[(\pi_0\mathbf{V})^{-1}].$$

To obtain further results one has, however, to assume that $\mathbf{V}$ is *cofibrant.*

9.2. Proposition. *If* $\mathbf{V} \in$ sO-**Cat** *is cofibrant, then the natural map* $\mathbf{V} \to \mathbf{V}[\mathbf{V}^{-1}]$ *induces a weak homotopy equivalence* $N\mathbf{V} \sim N(\mathbf{V}[\mathbf{V}^{-1}])$.

Proof. This follows by a diagonal argument from 7.6 and 3.7.

9.3. Proposition. *A map* $\mathbf{U} \to \mathbf{V} \in s\mathcal{O}\text{-}\mathbf{Cat}$ *between cofibrant objects induces a weak equivalence* $\mathbf{U}[\mathbf{U}^{-1}] \sim \mathbf{V}[\mathbf{V}^{-1}]$ *if and only if it induces a weak homotopy equivalence* $N\mathbf{U} \sim N\mathbf{V}$.

Proof. It is clearly no restriction to assume that *$N\mathbf{U}$ and $N\mathbf{V}$ are connected.* It then follows from 7.6 and the argument used in the proof of 5.5, that the simplicial sets $\mathbf{U}[\mathbf{U}^{-1}](X, Y)$ and $\mathbf{V}[\mathbf{V}^{-1}](X, Y)$ have the homotopy type of the loops on $N\mathbf{U}$ and $N\mathbf{V}$ respectively. The desired result now follows readily.

9.4. Corollary. *A weak equivalence* $\mathbf{U} \sim \mathbf{V} \in s\mathcal{O}\text{-}\mathbf{Cat}$ *between cofibrant objects induces a weak equivalence between their groupoid completions* $\mathbf{U}[\mathbf{U}^{-1}] \sim \mathbf{V}[\mathbf{V}^{-1}]$.

Less obvious is

9.5. Proposition. *Let* $\mathbf{V} \in s\mathcal{O}\text{-}\mathbf{Cat}$ *be cofibrant. Then the natural map* $\mathbf{V} \to \mathbf{V}[\mathbf{V}^{-1}]$ *is a weak equivalence if and only if* $\pi_0\mathbf{V}$ *is a groupoid.*

9.6. Corollary. *Let* $\mathbf{U}, \mathbf{V} \in s\mathcal{O}\text{-}\mathbf{Cat}$ *be cofibrant and such that* $\pi_0\mathbf{U}$ *and* $\pi_0\mathbf{V}$ *are groupoids. Then a map* $\mathbf{U} \to \mathbf{V} \in s\mathcal{O}\text{-}\mathbf{Cat}$ *is a weak equivalence if and only if the induced map* $N\mathbf{U} \to N\mathbf{V}$ *is a weak homotopy equivalence.*

Proof. The "only if" part is trivial since $\pi_0(\mathbf{V}[\mathbf{V}^{-1}])$ is a groupoid. To prove the "if" part, one first constructs the simplicial set $N(\mathbf{V}, \mathbf{V})$ which has as k-simplices the sequences

$$X_k \to X_{k-1} \to \cdots \to X_0 \to X_{-1}$$

of maps in $\mathbf{V}_k$ and which has the obvious face and degeneracy operators, and observe that the components of $N(\mathbf{V}, \mathbf{V})$ are contractible and are in 1–1 correspondence with the objects of $\mathbf{V}$. Next one considers the map $p : N(\mathbf{V}, \mathbf{V}) \to N\mathbf{V}$ obtained by omitting the "last map" and notes that, in view of the generalization of [9, Theorem 7.6] described in [9, Section 12], the fact that $\pi_0\mathbf{V}$ is a groupoid implies that the geometric realization of the map $p : N(\mathbf{V}, \mathbf{V}) \to N\mathbf{V}$ is a quasi-fibration. Hence the intersection of the counter image of a vertex $X_0 \in N\mathbf{V}$ with the components of $N(\mathbf{V}, \mathbf{V})$ have the homotopy type of the loops on $N\mathbf{V}$ at X_0. But it is easy to verify that these intersections consist exactly of the simplicial sets $\mathbf{V}(X_0, Y)$, where Y runs through the objects of $\mathbf{V}$. The desired result now follows readily from 9.2 and the argument used in the proof of 9.3.

9.7. Remark. According to [10, I, p. 4.2], Corollary 9.4 implies that *the groupoid completion functor* $s\mathcal{O}\text{-}\mathbf{Cat} \to s\mathcal{O}\text{-}\mathbf{Cat}$ *has a total left derived functor* $\text{Ho-}s\mathcal{O}\text{-}\mathbf{Cat} \to \text{Ho-}s\mathcal{O}\text{-}\mathbf{Cat}$ *given by* $\mathbf{V} \to \operatorname{diag} F_*\mathbf{V}[F_*\mathbf{V}^{-1}]$.

10. Localizations in sO-Cat

Finally we consider the notion of localization of maps in sO-**Cat** and obtain some of its basic properties, in particular Propositions 10.3 and 10.5, which immediately imply the not yet proven Lemmas 6.2 and 6.4.

10.1. Localization. Given a map $f: \mathbf{V} \to \mathbf{B} \in$ sO-**Cat**, the **V**-*localization* of **B** (the map f is understood) is the simplicial category $\mathbf{B}[\mathbf{V}^{-1}] \in$ sO-**Cat** in the pushout square in SO-**Cat**

$$\begin{array}{ccc} \mathbf{V} & \longrightarrow & \mathbf{V}[\mathbf{V}^{-1}] \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathbf{B} & \longrightarrow & \mathbf{B}[\mathbf{V}^{-1}] \end{array}$$

This definition readily implies

$$\pi_0(\mathbf{B}[\mathbf{V}^{-1}]) = (\pi_0\mathbf{B})[(\pi_0\mathbf{V})^{-1}] = (\pi_0\mathbf{B})[(\mathrm{im}\ \pi_0 f)^{-1}]$$

To obtain further results one has, however, to assume that **V** *and* **B** *are cofibrant and that f is a cofibration*, (such an f will be called a *strong cofibration*).

10.2. Proposition. *If* $\mathbf{V} \to \mathbf{B} \in$ sO-**Cat** *is a strong cofibration, then the natural map* $\mathbf{B} \to \mathbf{B}[\mathbf{V}^{-1}]$ *induces a weak homotopy equivalence* $N\mathbf{B} \sim N(\mathbf{B}[\mathbf{V}^{-1}])$.

Proof. This follows, by a diagonal argument, from 7.6 and 3.7.

10.3. Proposition. *Let*

$$\begin{array}{ccc} \mathbf{U} & \longrightarrow & \mathbf{V} \\ \downarrow & & \downarrow \\ \mathbf{A} & \longrightarrow & \mathbf{B} \end{array}$$

be a commutative diagram in sO-**Cat** *in which the horizontal maps are weak equivalences and the vertical maps are strong cofibrations. Then the induced map* $\mathbf{A}[\mathbf{U}^{-1}] \to \mathbf{B}[\mathbf{V}^{-1}]$ *is also a weak equivalence.*

Proof. This follows immediately from 9.4 and 8.1.

Less obvious is

10.4. Proposition. *Let* $\mathbf{V} \to \mathbf{B} \in$ sO-**Cat** *be a strong cofibration. Then the induced map* $\mathbf{B} \to \mathbf{B}[\mathbf{V}^{-1}]$ *is a weak equivalence if and only if every map of* $\pi_0\mathbf{B}$ *which is in the image of* $\pi_0\mathbf{V}$ *is invertible.*

Proof. The "only if" part is trivial. To prove the "if" part we first consider the special case that $\pi_0\mathbf{B}$ *is a groupoid*, in which case the desired result follows from 9.6 and 10.2. To prove the general case consider the subcategory $\mathbf{A} \subset \mathbf{B}$ consisting of all the maps which project to invertible elements of $\pi_0\mathbf{B}$ and construct a commutative diagram

$$\begin{tikzcd} \mathbf{V}_c \arrow[r] \arrow[d] & \mathbf{A}_c \arrow[r] \arrow[d] & \mathbf{B}_c \arrow[d] \\ \mathbf{V} \arrow[r] & \mathbf{A} \arrow[r, "\text{incl.}"] & \mathbf{B} \end{tikzcd}$$

in which

(i) the bottom map $\mathbf{V} \to \mathbf{B}$ is the given one,

(ii) the vertical maps are weak equivalences, and

(iii) the top maps are strong cofibrations.

In view of (ii) and 10.3 it then suffices to prove that the induced map $\mathbf{B}_c \to \mathbf{B}_c[\mathbf{V}_c^{-1}]$ is a weak equivalence, but this follows readily from the special case considered above ($\pi_0\mathbf{A}_c \approx \pi_0\mathbf{A}$ is a groupoid), 7.3 (ii) and the fact that (in view of (iii) the following diagram is a pushout

$$\begin{tikzcd} \mathbf{A}_c \arrow[r] \arrow[d] & \mathbf{A}_c[\mathbf{V}_c^{-1}] \arrow[d] \\ \mathbf{B}_c \arrow[r] & \mathbf{B}_c[\mathbf{V}_c^{-1}] \end{tikzcd}$$

Now we are ready for

10.5. Proposition. *Given a commutative diagram in* sO-**Cat**

$$\begin{tikzcd} \mathbf{U} \arrow[d] \arrow[dr, "f"] & \\ & \mathbf{B} \\ \mathbf{V} \arrow[ur, "g"'] & \end{tikzcd}$$

in which f and g are strong cofibrations, then the induced map $\mathbf{B}[\mathbf{U}^{-1}] \to \mathbf{B}[\mathbf{V}^{-1}]$ *is a weak equivalence if and only if*

$$\operatorname{im} \pi_0 g \subset \overline{\operatorname{im} \pi_0 f}$$

i.e. the image of $\pi_0\mathbf{V}$ in $\pi_0\mathbf{B}$ is contained in the closure (3.4) of the image of $\pi_0\mathbf{U}$ in $\pi_0\mathbf{B}$.

Proof. The "only if" part is trivial. To prove the "if" part of we may assume that the map $\mathbf{U} \to \mathbf{V}$ is also a strong cofibration. Then the diagram

$$\begin{tikzcd} \mathbf{V}[\mathbf{U}^{-1}] \arrow[r] \arrow[d] & \mathbf{V}[\mathbf{V}^{-1}] \arrow[d] \\ \mathbf{B}[\mathbf{U}^{-1}] \arrow[r] & \mathbf{B}[\mathbf{V}^{-1}] \end{tikzcd}$$

is a pushout in which the map on the left is a cofibration. Construct a commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r] \arrow[d] & \mathbf{V}[\mathbf{U}^{-1}] \arrow[d] \\ \mathbf{Y} \arrow[r] & \mathbf{B}[\mathbf{U}^{-1}] \end{tikzcd}$$

in which the horizontal maps are weak equivalences and the map on the left is a strong cofibration. In view of 9.6 and 10.2, the induced map $\mathbf{X}[\mathbf{X}^{-1}] \to \mathbf{V}[\mathbf{V}^{-1}]$ then is a weak equivalence and hence (8.1) so is the induced map $\mathbf{Y}[\mathbf{X}^{-1}] \to \mathbf{B}[\mathbf{V}^{-1}]$. It thus remains to show that the induced map $\mathbf{Y} \to \mathbf{Y}[\mathbf{X}^{-1}]$ is a weak equivalence, but this follows immediately from the fact that the map $\mathbf{X} \to \mathbf{Y}$ satisfies all the conditions of 10.4.

10.6. Remark. Let sO-**Cat**′ denote the category of maps in sO-**Cat**. Then [2] the closed model category structure on sO-**Cat** induces one on sO-**Cat**′ in which the weak equivalences are the obvious maps and the cofibrant objects are the strong cofibrations of sO-**Cat**. According to [10, I, p. 4.2] Proposition 10.3 then implies that *the localization functor* sO-**Cat**′ $\to$ sO-**Cat** *has a total left derived functor* Ho-sO-**Cat**′ $\to$ Ho-sO-**Cat**.

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