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Singular functors and realization functors*

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SUMMARY.

In [6] Quillen showed that the singular functor and the realization functor have certain properties which imply the equivalence of the weak homotopy theory of topological spaces with the homotopy theory of simplicial sets. The aim of this note is to generalize this result and to show that one can, in essentially the same manner, establish the equivalence of other homotopy theories (e.g., the *equivariant* homotopy theories) with homotopy theories of *simplicial diagrams of simplicial sets.* Applications to equivariant homotopy will be given in [3] and [4].

§ 1. THE MAIN RESULT

In this section we outline our main results (1.1 and 1.5) and discuss their application to equivariant homotopy and to simplicial diagrams of simplicial sets. The precise statements of the theorems and their proofs will be given in § 2 and § 3.

Our first result concerns

1.1 MODEL CATEGORY STRUCTURES. Let $\mathbf{M}$ be a simplicial category and let $\{O_e\}_{e\in E}$ be a set of objects of $\mathbf{M}$ which satisfies the conditions of 2.1, i.e., *is a set of orbits for* $\mathbf{M}$. Then (2.2) $\mathbf{M}$ *admits a closed simplicial model category structure* in which the simplicial structure is the given one and in which a map $X\to Y\in\mathbf{M}$ is a weak equivalence or a fibration iff, for every element $e\in E$, the

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induced map of function complexes $\hom(O_e, X) \to \hom(O_e, Y)$ is a weak equivalence or a fibration of simplicial sets.

Two interesting special cases are:

1.2 EQUIVARIANT HOMOTOPY THEORIES. Let G be an arbitrary but fixed topological group and let $\mathbf{M}$ be the category of *topological spaces with a left G-action*. Then $\mathbf{M}$ admits an obvious simplicial structure in which, for every two objects $X, Y \in \mathbf{M}$, the function complex $\hom(X, Y)$ is the simplicial set which has as n-simplices the maps $X \times |\Delta[n]| \to Y \in \mathbf{M}$ (where $|\Delta[n]|$ denotes the realization of the standard n-simplex with the trivial G-action). Clearly, for every subgroup $G_a \subset G$, the obvious left G-action on the left coset space G/G_a turns G/G_a into an object of $\mathbf{M}$ with the property that, *for every object $X \in \mathbf{M}$, the simplicial set* $\hom(G/G_a, X)$ *is naturally isomorphic to the singular complex* Sing X^a *of the fixed point set X^a of* G_a. Given a set $\{G_a\}_{a \in A}$ of subgroups of G, it is now not difficult to verify that the set $\{G/G_a\}_{a \in A}$ of G-spaces is a set of orbits for $\mathbf{M}$ and hence *the category* $\mathbf{M}$ *admits a closed simplicial model category structure* in which the simplicial structure is the above one and in which a map $X \to Y \in \mathbf{M}$ is a weak equivalence or a fibration iff, for every element $a \in A$, the induced map of fixed point sets $X^a \to Y^a$ is a weak homotopy equivalence or a Serre fibration.

No separation axioms were assumed. However all the above statements remain valid if, for instance, all topological spaces and topological groups are assumed to be *compactly generated* (*though not necessarily Hausdorff*) or *singularly generated* (i.e., the topology is the identification topology obtained from the realization of the singular complex).

1.3 SIMPLICIAL DIAGRAMS OF SIMPLICIAL SETS. Let $\mathbf{C}$ be a small simplicial category, let $\mathbf{S}$ be the category of simplicial sets and let $\mathbf{S}^{\mathbf{C}}$ denote the category of *$\mathbf{C}$-diagrams of simplicial sets* (which has as objects the simplicial functors $\mathbf{C} \to \mathbf{S}$ and as maps the natural transformations between them). The simplicial structure on $\mathbf{S}$ induces a simplicial structure on $\mathbf{S}^{\mathbf{C}}$ and it is not difficult to verify that, *for every diagram $X \in \mathbf{S}^{\mathbf{C}}$ and every object $C \in \mathbf{C}$, there is a natural isomorphism* $\hom(\hom(C, -), X) \approx XC$. From this one readily deduces that the $\mathbf{C}$-diagrams $\{\hom(C, -)\}$, where C runs through the objects of $\mathbf{C}$, form a set of orbits for $\mathbf{S}^{\mathbf{C}}$ and *the category $\mathbf{S}^{\mathbf{C}}$ thus admits a closed simplicial model category structure* in which the simplicial structure is as above and in which a map $X \to Y \in \mathbf{S}^{\mathbf{C}}$ is a weak equivalence or a fibration iff, for every object $C \in \mathbf{C}$, the induced map $XC \to YC \in \mathbf{S}$ is so.

1.4 REMARK. E. Dror recently observed that suitable larger sets of orbits give rise to other and interesting closed simplicial model category structures on $\mathbf{S}^{\mathbf{C}}$ with stronger notions of weak equivalences.

Now we turn to

1.5 SINGULAR FUNCTORS AND REALIZATION FUNCTORS. Let again $\mathbf{M}$ be a simplicial category and $\{O_e\}_{e \in E}$ a set of orbits for $\mathbf{M}$ and let $\mathbf{O} \subset \mathbf{M}$ be the resulting orbit category, i.e., the full simplicial subcategory spanned by the O_e ($e \in E$). Then (3.1) *the obvious singular functor* $\hom(\mathbf{O}, -): \mathbf{M} \to \mathbf{S}^{\mathbf{O}^{op}}$ *has as left adjoint a realization functor* $\mathbf{O} \otimes : \mathbf{S}^{\mathbf{O}^{op}} \to \mathbf{M}$, and this pair of adjoint functors has properties which imply the *equivalence of the homotopy theory of* $\mathbf{M}$ *with the homotopy* theory of $\mathbf{S}^{\mathbf{O}^{op}}$.

Applying this to 1.3 and 1.2 one gets

1.6 SIMPLICIAL DIAGRAMS OF SIMPLICIAL SETS. Here nothing new happens as $\mathbf{O}^{op} \approx \mathbf{C}$ and the resulting singular and realization functors both coincide with the identity functor of $\mathbf{S}^{\mathbf{C}}$.

1.7 EQUIVARIANT HOMOTOPY THEORIES. Our results imply that every equivariant homotopy theory is equivalent to a theory of simplicial diagrams of simplicial sets. Some special cases are:

(i) THE TRIVIAL CASE. If $G = 1$, the trivial group, then the singular and realization functors are just the usual ones between the categories of topological spaces and simplicial sets.

(ii) THE CASE OF A SINGLE SUBGROUP If the set $\{G_{a \in A}\}_A$ consists of only one subgroup $G_a \subset G$, then the functor $\hom(\mathbf{O}, -)$ assigns to every G-space $X \in \mathbf{M}$ the singular complex Sing X^a of the fixed point set X^a of G_a, with an action of the simplicial monoid $\mathrm{Sing}(N'G_a/G_a)$, where $N'G_a \subset G$ denotes the submonoid which consists of the elements $h \in G$ such that $h^{-1}G_a h \subset G_a$. Thus *this equivariant homotopy theory* is equivalent to *the homotopy theory of simplicial sets with a* $\mathrm{Sing}(N'G_a/G_a)$*-action.*

If G_a is a *normal* subgroup of G, then clearly $N'G_a = G$ and hence $N'G_a/G_a = G/G_a$.

(iii) THE CASE OF TWO SUBGROUPS. If there are only two subgroups G_a and G_b, then the orbit category $\mathbf{O}$ has two objects $O_a = G/G_a$ and $O_b = G/G_b$ and its function complexes are given by the formulas

$$\hom(O_a, O_a) = \mathrm{Sing}(N'G_a/G_a) \qquad \hom(O_b, O_a) = \mathrm{Sing}(N'(G_b, G_a)/G_a)$$

$$\hom(O_b, O_b) = \mathrm{Sing}(N'G_b/G_b) \qquad \hom(O_a, O_b) = \mathrm{Sing}(N'(G_a, G_b)/G_b)$$

where $N'(G_a, G_b) \subset G$ denotes the sub*space* which consists of the elements $k \in G$ such that $k^{-1}G_a k \subset G_b$.

If G_a and G_b are both *normal* subgroups of G, then $N'(G_a, G_b) = G$ if $G_a \subset G_b$ and empty otherwise.

The general equivariant case is essentially the same as the one described in (iii).

The relationship between equivariant homotopy theory and the theory of diagrams was also studied by Elmendorf [5], though from a slightly different point of view.

In theorem 2.2 below we give sufficient conditions for a simplicial category to admit a closed simplicial model category structure. In order to simplify its formulation we first introduce the notion of

2.1 SETS OR ORBITS. Let **M** be a simplicial category satisfying axioms *MO* and *SMO* of Quillen [6], i.e., **M** is closed under finite direct and inverse limits and $X \otimes K$ and X^K exist for every object $X \in \mathbf{M}$ and every finite simplicial set K (a simplicial set is called finite if it has only a finite number of non-degenerate simplices). A set $\{O_e\}_{e \in E}$ of objects of **M** then is said to be *a set of orbits for* **M** if, in addition, the following four axioms hold, the middle two of which state that *the functors* $\hom(O_e, -)$: $\mathbf{M} \to \mathbf{S}$ *commute, up to homotopy, with certain direct limits,* while the last axiom permits the use of *Quillen's small object argument* [6, II, 3.4].

Q0. **M** is closed under arbitrary direct limits.

Q1. If

$$\begin{array}{ccc} O_e \otimes K & \longrightarrow & O_e \otimes L \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & X_{a+1} \end{array}$$

is a push out diagram in **M** in which L is a finite simplicial set, K is a subcomplex of L and $e \in E$, then, for every $e' \in E$, the induced diagram in **S**

$$\begin{array}{ccc} \hom(O_{e'}, O_e \otimes K) & \longrightarrow & \hom(O_{e'}, O_e \otimes L) \\ \downarrow & & \downarrow \\ \hom(O_{e'}, X_a) & \longrightarrow & \hom(O_{e'}, X_{a+1}) \end{array}$$

is up to homotopy (i.e., up to a weak equivalence) a push out diagram.

Q2. If $X_1 \to \cdots \to X_a \to X_{a+1} \to \cdots$ is a (possibly transfinite) sequence of objects and maps in **M** such that each map $X_a \to X_{a+1}$ is induced as in Q1 and such that, for every limit ordinal b involved, one has $X_b = \varinjlim^{a<b} X_a$, then, for every $e \in E$, the induced map

$$\varinjlim{}^a \hom(O_e, X_a) \to \hom(O_e, \varinjlim{}^a X_a) \in \mathbf{S}$$

is a weak equivalence.

Q3. There is a limit ordinal c such that, for every sequence $X_1 \to \cdots \to \to X_a \to X_{a+1} \to \cdots$ as in Q2 which is indexed by the ordinals $<c$ and for every $e \in E$, one has:

$$\varinjlim{}^a \hom(O_e, X_a) = \hom(O_e, \lim{}^a X_a).$$

Now we can fomulate

2.2 THEOREM. *Let* **M** *be a simplicial category satisfying axioms MO and SMO of Quillen* [6] *and let* $\{O_e\}_{e \in E}$ *be a set of orbits for* **M**. *Then* **M** *admits a closed simplicial model category structure in which the simplicial structure is the given one, in which a map* $X \to Y \in \mathbf{M}$ *is a weak equivalence or a fibration, iff, for every element* $e \in E$, *the induced map* $\hom(O_e, X) \to \hom(O_e, Y) \in \mathbf{S}$ *is so and in which the cofibrations are the retracts of the maps* $X \to Y$ *which admit (possibly transfinite) factorizations*

$$X = X_1 \to \cdots \to X_a \to X_{a+1} \to \cdots \to \lim{}^a X_a = Y$$

in which each map $X_a \to X_{a+1}$ is induced as in Q1 and in which, for every limit ordinal b involved, one has $X_b = \lim^{a<b} X_a$.

PROOF. In view of [6, Ch. II, § 2, prop. 3], it suffices to prove that **M** is a closed model category (i.e., satisfies axioms CM1–5 of [7, p. 233] and in addition satisfies axiom SM7(a) of [6, Ch. 22, § 2]. Of these axioms SM7(a), CM1, CM2, CM3, and CM4(i) are obvious, and it thus remains to verify axioms CM4(ii), CM5(i), and CM5(ii), which we will do using the notation and terminology of [6].

Verification of CM5(i). The trivial fibrations in **M** are characterized by the right lifting property with respect to the maps

$$O_e \otimes \dot{\Delta}[n] \to O_e \otimes \Delta[n] \qquad \text{where } e \in E \text{ and } n \geq 0$$

and one can thus verify CM5(i) by the small object argument of [6, Ch. II, 3.4].

Verification of CM5(ii). The fibrations in **M** are characterized by the right lifting property with respect to the weak equivalences

$$O_e \otimes V[n,k] \to O_e \otimes \Delta[n] \qquad \text{where } e \in E \text{ and } n \geq k \geq 0$$

and axiom CM5(ii) can therefore be verified by a similar small object argument. (The fact that these maps are weak equivalences follows by induction from the glueing axiom (Q1) and the observation that, for any $X \in \mathbf{M}$ and $n \geq 0$, the natural map $p: X \otimes \Delta[n] \to X$ is a weak equivalence. This in turn is proved by showing that, if $i_0: X \to X \otimes \Delta[n]$ is the "zero vertex inclusion," then pi_0 is the identity map of X and $i_0 p$ is connected to the identity map of $X \otimes \Delta[n]$ by a natural 1-simplex of $\hom(X \otimes \Delta[n], X \otimes \Delta[n])$).

Verification of CM4(ii). By the above arguments a trivial cofibration $X \to Y$ admits a factorization $X \to Y' \to Y$ in which (by construction) the map $X \to Y'$ is a trivial cofibration which has the left lifting property with respect to the fibrations and $Y' \to Y$ is a (necessarily trivial) fibration. Moreover the map $X \to Y$ is readily seen to be a retract of the map $X \to Y'$, and it thus also has the left lifting property with respect to the fibrations.

§ 3. SINGULAR FUNCTORS AND REALIZATION FUNCTORS

We now show that the homotopy theories produced by theorem 2.2 are always equivalent to homotopy theories of simplicial diagrams of simplicial sets. More precisely

3.1 THEOREM. *Let* $\mathbf{M}$ *be a simplicial category satisfying axioms MO and SMO of Quillen* [6]. *Let* $\{O_e\}_{e\in E}$ *be a set of orbits for* $\mathbf{M}$ *and let* $\mathbf{O} \subset M$ *be the resulting orbit category, i.e., the full simplicial subcategory spanned by the* O_e $(e \in E)$. *Then the "singular functor":*

$$\hom(\mathbf{O}, -) : \mathbf{M} \to \mathbf{S}^{\mathbf{O}^{op}}$$

has a left adjoint, the "realization functor":

$$\mathbf{O}\otimes : \mathbf{C}^{\mathbf{O}^{op}} \to \mathbf{M}.$$

Moreover, in the model category structures of 1.3 *and* 2.2:

(i) *the functor* $\hom(\mathbf{O}, -)$ *preserves fibrations and weak equivalences between fibrant objects,*

(ii) *the functor* $\mathbf{O}\otimes$ *preserves cofibrations and weak equivalences between cofibrant objects, and*

(iii) *for every cofibrant object* $X \in \mathbf{S}^{\mathbf{O}^{op}}$ *and every fibrant object* $Y \in \mathbf{M}$, *a map* $\mathbf{O}\otimes X \to Y \in \mathbf{M}$ *is a weak equivalence iff its adjoint* $X \to \hom(\mathbf{O}, Y) \in \mathbf{S}^{\mathbf{O}^{op}}$ *is so,*

and hence $\mathbf{M}$ and $\mathbf{S}^{\mathbf{O}^{op}}$ have equivalent homotopy theories in the sense that they have the same simplicial homotopy categories [3, § 5].

3.2 COROLLARY. *The full simplicial subcategories of* $\mathbf{M}$ *and* $\mathbf{S}^{\mathbf{O}^{op}}$ *spanned by the objects which are both fibrant and cofibrant are weakly equivalent in the sense of* [2, § 2].

PROOF OF THEOREM 3.1. Given an object $X \in \mathbf{S}^{\mathbf{O}^{op}}$, let $\mathbf{O}\otimes X$ be the direct limit of the diagram in $\mathbf{M}$ which contains, for every object $O_e \in \mathbf{O}$, the object $O_e \otimes XO_e$ (which is readily verified to exist) and, for every pair of objects $O_e, O_{e'} \in \mathbf{O}$, integer $n \geq 0$ and map $g: O_e \otimes \Delta[n] \to O_{e'} \in \mathbf{M}$, the object $O_e \otimes (\Delta[n] \times XO_{e'})$ and the pair of maps:

$$O_e \otimes XO_e \xrightarrow{O_e \otimes X_g} O_e \otimes (\Delta[n] \times XO_{e'}) = (O_e \otimes \Delta[n]) \otimes XO_{e'} \xrightarrow{g \otimes XO_{e'}} O_{e'} \otimes XO_{e'}.$$

A straightforward calculation then yields that $\mathbf{O}\otimes$ is actually a functor and is in fact a left-adjoint of the functor $\hom(\mathbf{O}, -)$.

Part (i) now follows from the fact that the functor hom($\mathbf{O}$, $-$) preserves fibrations and weak equivalences; the adjointness therefore implies that the functor $\mathbf{O}\otimes$ preserves cofibrations as well as trivial cofibrations and hence [1, 1.2 and 1.3] weak equivalences between cofibrant objects.

Finally to prove (iii), one notes that a map $Y \to Y' \in \mathbf{M}$ is a weak equivalence iff the induced map $\hom(\mathbf{O}, Y) \to \hom(\mathbf{O}, Y') \to \mathbf{S}^{\mathbf{O}^{op}}$ is so, and it therefore suffices to show that, for every cofibrant object $X \in \mathbf{S}^{\mathbf{O}^{op}}$, the adjunction map $X \to \hom(\mathbf{O}, \mathbf{O} \otimes X)$ is a weak equivalence. But this follows readily from the fact that

(i) for every object $O_e \in \mathbf{O}$, there is an obvious isomorphism

$$O \otimes \hom(\mathbf{O}, O_e) \approx O_e$$

and hence an isomorphism:

$$\hom(\mathbf{O}, \mathbf{O} \otimes \hom(\mathbf{O}, O_e)) \approx \hom(\mathbf{O}, O_e)$$

(ii) this readily implies that, for every object $O_e \in \mathbf{O}$ and integer $n \geq 0$, the adjunction map

$$\hom(\mathbf{O}, O_e \otimes \Delta[n]) \to \hom(\mathbf{O}, \mathbf{O} \otimes \hom(\mathbf{O}, \mathbf{O}_e \otimes \Delta[n]))$$

is a weak equivalence,

(iii) every cofibrant object of $\mathbf{S}^{\mathbf{O}^{op}}$ is a retract of the direct limit of a (possibly transfinite) sequence of map

$$\phi = X_1 \to \cdots \to X_a \to X_{a+1} \to \cdots$$

with the obvious (see 2.2) properties, and

(iv) the functor $\mathbf{O}\otimes$ preserves push outs on the nose while the functor hom($\mathbf{O}$, $-$) preserves the needed push outs up to homotopy.

BIBLIOGRAPHY

1. Brown, K.S. – Abstract homotopy theory and generalized sheaf cohomology, Trans. AMS **186**, 419–458 (1974).
2. Dwyer, W.G. and D.M. Kan – Function complexes in homotopical algebra, Topology **19**, 427–440 (1980).
3. Dwyer, W.G. and D.M. Kan – Equivariant homotopy classification, to appear.
4. Dwyer, W.G. and D.M. Kan – Reducing equivariant homotopy theory to the theory of fibrations, to appear.
5. Elmendorf, A.D. – Systems of fixed point sets, Trans. AMS **277**, 275–284 (1983).
6. Quillen, D.G. – Homotopical algebra, Lect. Notes in Math. **43**, Springer (1967).
7. Quillen, D.G. – Rational homotopy theory, Ann. of Math. **90**, 205–295 (1969).