

# THE SPACES THAT DEFINE ALGEBRAIC $K$ -THEORY

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ABSTRACT. We characterize spaces  $W$  such that the  $W$ -nullification functor  $P_W$ , applied to any  $\mathrm{BGL}(R)$ , gives  $\mathrm{BGL}(R)^+$ .

Let  $X$  be a pointed space and  $X^+$  the Quillen plus-construction on  $X$  with respect to the maximal perfect subgroup of  $\pi_1 X$ . If  $W$  is a pointed CW-complex, let  $P_W(X)$  denote the  $W$ -nullification of  $X$ . (Recall that  $P_W X$  is up to homotopy the initial space  $Y$  under  $X$  which is  $W$ -null in the sense that the pointed mapping space  $\mathrm{Map}_*(W, Y)$  is weakly contractible.) For some time it has been known that there exist universal spaces  $W$  such that for any  $X$ ,  $P_W(X)$  is equivalent to  $X^+$ . We will say that such a space  $W$  *defines the plus-construction*.

Given that the plus-construction was originally applied to spaces of the form  $\mathrm{BGL}(R)$  in order to construct the higher algebraic  $K$ -groups of a ring  $R$ , it seems natural to consider spaces  $W$  with the property that for any discrete, associative ring  $R$  with unit,  $P_W \mathrm{BGL}(R)$  is equivalent to  $\mathrm{BGL}(R)^+$ . We will say that such a  $W$  *defines algebraic  $K$ -theory*. For then one can use  $W$  to define the algebraic  $K$ -groups of a ring  $R$  as

$$K_i(R) = \begin{cases} \pi_i(P_W \mathrm{BGL}(R)) & i \geq 1 \\ \pi_1(P_W \mathrm{BGL}(S^{1-i}R)) & i \leq 0 \end{cases}$$

where  $S$  denotes suspension of rings. The question of characterizing such  $W$  was first raised in [2], and solved for finite-dimensional  $W$  in [1]. Of course any  $W$  which defines the plus-construction also defines algebraic  $K$ -theory, but it turns out that some other spaces can define algebraic  $K$ -theory as well.

By choosing special test rings  $R$ , we are able in this note to give a complete characterization of the spaces  $W$  which define algebraic  $K$ -theory. Recall that a space is said to be *acyclic* if its reduced integral homology vanishes.

**Theorem 1.** *A CW-complex  $W$  defines algebraic  $K$ -theory if and only if both*

- (i)  *$W$  is acyclic, and*
- (ii) *there is a nontrivial homomorphism  $\pi_1 W \rightarrow \mathrm{GL}(\mathbb{Z})$ .*

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**Remarks.**

(a) Examples of spaces satisfying (i) but not (ii) are the classifying spaces of acyclic groups that lack a nontrivial integral representation, for instance the McLain group  $M(\mathbb{Q}, \mathbb{F}_p)$ , Higman's four-generator four-relator group, binate groups, and many acyclic automorphism groups. See [1, Example 5.6] for further discussion.

(b) In (ii) it is not necessary to have a nontrivial finite-dimensional representation. For example, the fundamental group of the acyclic fibre  $A$  of  $\mathrm{BGL}(\mathbb{Z}) \rightarrow \mathrm{BGL}(\mathbb{Z})^+$ , namely the Steinberg group  $\mathrm{St}(\mathbb{Z})$ , maps onto the group  $\mathrm{E}(\mathbb{Z}) \subset \mathrm{GL}(\mathbb{Z})$  generated by elementary matrices, but has no nontrivial homomorphic image in any  $\mathrm{GL}_n(\mathbb{Z})$ . To check this last assertion it suffices to show that  $\mathrm{St}(\mathbb{Z})$  has no nontrivial finite quotients (this follows from the fact that any nontrivial element of  $\mathrm{GL}_n(\mathbb{Z})$  reduces to a nontrivial element of some  $\mathrm{GL}_n(\mathbb{Z}/p)$ ). Given a homomorphism from  $\mathrm{St}(\mathbb{Z})$  to a finite group, write  $\bar{x}_{ij}$  for the image of the standard generator  $x_{ij}$  (the notation requires  $i \neq j$ ). If some  $\bar{x}_{ij} \neq 1$ , then, since  $\bar{x}_{ij} = [\bar{x}_{ik}, \bar{x}_{kj}]$  for all possible  $k$  (a relation that holds in  $\mathrm{St}(\mathbb{Z})$  itself), it must be that  $\bar{x}_{ij} \neq 1$  for all possible  $j$ . Then by the finiteness of the codomain we can find distinct  $j, k$  with  $\bar{x}_{ij} = \bar{x}_{ik}$ . Hence,  $\bar{x}_{ij} = [\bar{x}_{ij}, \bar{x}_{kj}] = 1$ , a relation which immediately implies  $\bar{x}_{ij} = 1$  and gives the desired contradiction.

(c) It follows from (b) that  $A$  is an example of a space that defines algebraic  $K$ -theory but does not define the plus-construction. This is because if  $\pi$  is any finite group then

$$[A, B\pi] = \mathrm{Hom}(\mathrm{St}(\mathbb{Z}), \pi)$$

is trivial, so that  $B\pi$  is  $A$ -null and  $P_A B\pi = B\pi$ .

The sufficiency of the conditions in the theorem has been proved elsewhere, and so we discuss it only briefly here. First, concerning condition (i), it is shown in [2, Theorem 3.3] that if  $W$  is an acyclic space, the  $W$ -nullification  $X \mapsto P_W X$  corresponds to the plus-construction  $X \mapsto X_{T(W, X)}^+$  with respect to a certain perfect normal subgroup  $T(W, X)$  of  $\pi_1(X)$ . Furthermore, Theorem 5.1 and Corollary 5.5 of [1] reveal that, if  $W$  is acyclic or more generally if  $H_1(W; \mathbb{Z}) = 0$ , then  $T(W, \mathrm{BGL}(R)) = \mathrm{E}(R)$  for all rings  $R$  if and only if condition (ii) above holds. (When (ii) fails,  $T(W, \mathrm{BGL}(\mathbb{Z}[\pi_1 W])) = \mathrm{E}(\mathbb{Z}[\pi_1 W], \mathrm{Aug}_{\mathbb{Z}}(\pi_1 W))$ , where this last is the relative elementary matrix group with respect to the augmentation ideal in the integral group ring.)

Turning now to necessity of the two conditions, we first note the simplification of Lemma 5.5 of [2]: by considering the case  $R = \mathbb{C}$  it is shown there that if  $W$  defines algebraic  $K$ -theory then  $H_1(W; \mathbb{Z}) = 0$ .

The kernel of the natural map  $\mathrm{GL}(R) \rightarrow \pi_1 \mathrm{BGL}(R)^+$  is  $T(W, \mathrm{BGL}(R))$  [1]; it follows that  $T(W, \mathrm{BGL}(R)) = \mathrm{E}(R)$  and so, by the above result of [1], that  $W$  satisfies condition (ii). To finish the proof of the theorem it is therefore enough to show that  $W$  is acyclic. We will deduce this from the fact that if  $W$  defines algebraic  $K$ -theory then every space of the form  $\mathrm{BGL}(R)^+$  is  $W$ -null.

We will test  $W$ -nullity of  $\mathrm{BGL}(R)^+$  on rings  $R$  related to rings of dual numbers. Recall that the ring of dual numbers  $R[\varepsilon]$  is the truncated polynomial ring  $R[t]/(t^2)$ . The tangent space to algebraic  $K$ -theory at  $R$  is defined to be the homotopy fibre of the map from  $\mathrm{BGL}(R[\varepsilon])^+$  to  $\mathrm{BGL}(R)^+$  induced by  $\varepsilon \mapsto 0$ . When  $R = \mathbb{F}_p$  ( $p$  any prime) we denote this homotopy fibre by  $F_p$ . The space  $F_1$  is defined to be the corresponding fibre in the case  $R = \mathbb{Z}$ . We also write  $F_0$  for the homotopy fibre of the map from  $\mathrm{BGL}(\mathbb{Z} + \mathbb{Q}\varepsilon)^+$  to  $\mathrm{BGL}(\mathbb{Z})^+$  induced by  $\varepsilon \mapsto 0$ , with  $\mathbb{Z} + \mathbb{Q}\varepsilon$  considered as a subring of  $\mathbb{Q}[\varepsilon]$ . As the fibre of a map between  $W$ -null spaces, each such  $F_n$  must also be  $W$ -null.

Our first aim is to show that  $W$  is rationally acyclic. To do this, we prove that the natural map  $\gamma : F_1 \rightarrow F_0$  is rationalization. For any  $R$ , let  $\mathrm{M}(R)$  be the union of additive matrix groups  $\mathrm{M}_n(R)$ , isomorphic via  $M \mapsto I + M\varepsilon$  to the kernel of the map  $\mathrm{GL}(R[\varepsilon]) \rightarrow \mathrm{GL}(R)$  induced by the projection  $R[\varepsilon] \rightarrow R$ . Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{BM}(\mathbb{Z}) & \longrightarrow & \mathrm{BGL}(\mathbb{Z}[\varepsilon]) & \xrightarrow{p_1} & \mathrm{BGL}(\mathbb{Z}) \\ \alpha \downarrow & & \beta \downarrow & & \mathrm{id} \downarrow \\ \mathrm{BM}(\mathbb{Q}) & \longrightarrow & \mathrm{BGL}(\mathbb{Z} + \mathbb{Q}\varepsilon) & \xrightarrow{p_0} & \mathrm{BGL}(\mathbb{Z}) \end{array}$$

Since  $\mathrm{M}(\mathbb{Z})$  is just a copy of the countably infinite free abelian group and  $\mathrm{M}(\mathbb{Q}) = \mathrm{M}(\mathbb{Z}) \otimes \mathbb{Q}$ ,  $\alpha$  induces an isomorphism on rational homology, and, by the Serre spectral sequence, so does  $\beta$ . Since the  $\mathbb{F}_p$  homology of  $\mathrm{BM}(\mathbb{Q})$  is trivial (for any  $p$ ), it follows from the Serre spectral sequence again that  $p_0$  induces an isomorphism on  $\mathbb{F}_p$ -homology. Consequently, the induced map  $\beta^+$  is an isomorphism on rational homology, and the map  $p_0^+$  an isomorphism on  $\mathbb{F}_p$ -homology. Each map  $p_i$  is split, making  $p_i^+$  a split infinite loop space map. This implies that the fibration  $p_i^+$  is a product fibration of infinite loop spaces, and so there are parallel product decompositions

$$\begin{array}{ccc} \mathrm{BGL}(\mathbb{Z}[\varepsilon])^+ & \xrightarrow{\sim} & F_1 \times \mathrm{BGL}(\mathbb{Z})^+ \\ \beta^+ \downarrow & & \gamma \times \mathrm{id} \downarrow \\ \mathrm{BGL}(\mathbb{Z} + \mathbb{Q}\varepsilon)^+ & \xrightarrow{\sim} & F_0 \times \mathrm{BGL}(\mathbb{Z})^+ \end{array}$$

Clearly, then,  $\gamma$  induces an isomorphism on rational homology, and the reduced  $\mathbb{F}_p$ -homology of  $F_0$  is trivial. This last fact implies that the reduced integral homology groups of  $F_0$  are rational vector spaces. Since  $F_0$  is an infinite loop space, its homotopy groups are also rational vector spaces, and so  $\gamma$  is a rationalization map.

From the computation in [5] of the torsion-free rank of the homotopy groups of  $F_1$ , we deduce that the homotopy groups of  $F_0$  are  $\mathbb{Q}$  in odd dimensions, zero otherwise. Now, again, as the fibre of the infinite loop space map  $p_0^+$ ,  $F_0$  is itself an infinite loop space. Hence, by for example [4, Theorem 8.7],  $F_0$  is a generalized Eilenberg-Mac Lane space, the product of all spaces  $K(\mathbb{Q}, 2j - 1)$ . Since  $F_0$  is  $W$ -null, so is each  $K(\mathbb{Q}, 2j - 1)$ , giving  $W$  trivial rational cohomology in all dimensions (for each  $[W, \Omega K(\mathbb{Q}, 2j - 1)]$  is also null). It follows by universal coefficients that all the groups  $\text{Hom}(\tilde{H}_i(W; \mathbb{Z}), \mathbb{Q})$  are trivial and hence that the groups  $\tilde{H}_i(W; \mathbb{Z})$  are torsion. (Note that we avoid those universal coefficient theorems that require hypotheses of finite generation.)

To study this torsion, we use the fact that for each prime  $p$  the tangent space  $F_p$  is  $W$ -null. Now  $F_p$  is shown in [3, Sections 1, 9] to be a GEM, with all its even-dimensional homotopy groups zero. For each  $j \geq 1$ ,  $\pi_{2j-1}(F_p)$  is a (nontrivial) finite abelian  $p$ -group of exponent  $p^n$ , where  $n = 1$  when  $p = 2$  and  $n = 1 + [\log_p(2j - 1)]$  otherwise.

We therefore have, whenever  $1 \leq i \leq 2j - 1$ , the triviality of

$$[\Sigma^{2j-1-i}W, K(\pi_{2j-1}(F_p), 2j - 1)] = \tilde{H}^i(W; \pi_{2j-1}(F_p)),$$

and thus of its direct summands

$$\text{Hom}(\tilde{H}_i(W; \mathbb{Z}), \pi_{2j-1}(F_p)) \quad \text{and} \quad \text{Ext}(\tilde{H}_{i-1}(W; \mathbb{Z}), \pi_{2j-1}(F_p)) .$$

Since  $\pi_{2j-1}(F_p)$  is a nontrivial finite  $p$ -group, the vanishing of ‘‘Hom’’ implies that multiplication by  $p$  on  $\tilde{H}_i(W; \mathbb{Z})$  is surjective. This is true for all  $i$ , because  $j$  is arbitrary. Suppose now that  $A$  is an abelian group such that multiplication by  $p$  on  $A$  is surjective, that  $\pi$  is a nontrivial finite abelian  $p$ -group, and that  $\text{Ext}(A, \pi) = 0$ . Consideration of the Hom-Ext sequence for the exact sequence

$$0 \rightarrow {}_pA \rightarrow A \xrightarrow{p} A \rightarrow 0$$

shows that  $\text{Hom}({}_pA, \pi) = 0$ , so that  ${}_pA = 0$  and hence that multiplication by  $p$  on  $A$  is an isomorphism. We conclude that for all primes  $p$ , multiplication by  $p$  on  $\tilde{H}_*(W; \mathbb{Z})$  is an isomorphism. Since as above these reduced integral homology groups are torsion,  $W$  is acyclic after all.

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