ON THE K-THEORY SPECTRUM OF A RING OF ALGEBRAIC INTEGERS

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§1. Introduction

Suppose that F is a number field (i.e. a finite algebraic extension of the field $\mathbb{Q}$ of rational numbers) and that $\mathcal{O}_F$ is the ring of algebraic integers in F. One of the most fascinating and apparently difficult problems in algebraic K-theory is to compute the groups $K_i\mathcal{O}_F$. These groups were shown to be finitely generated by Quillen [36] and their ranks were calculated by Borel [7]. Lichtenbaum and Quillen have formulated a very explicit conjecture about what the groups should be [37]. Nevertheless, there is not a single number field F for which $K_i\mathcal{O}_F$ is known for any $i \geq 5$. For most fields F, these groups are unknown for $i \geq 3$.

The groups $K_i\mathcal{O}_F$ are the homotopy groups of a spectrum $K\mathcal{O}_F$ ("spectrum" in the sense of algebraic topology [1]). In this paper, rather than concentrating on the individual groups $K_i\mathcal{O}_F$, we study the entire spectrum $K\mathcal{O}_F$ from the viewpoint of stable homotopy theory. For technical reasons, it is actually more convenient to single out a rational prime number ℓ and study instead the spectrum KR_F, where $R_F = \mathcal{O}_F[1/\ell]$ is the ring of algebraic ℓ-integers in F. "At ℓ", that is, after localizing or completing at ℓ, the spectra $K\mathcal{O}_F$ and KR_F are almost the same (cf. [34] and [35, p. 113]).

The main results. We will begin by describing the two basic results in this paper. Assume that ℓ is an *odd* prime. If X is a space or spectrum, let $\hat{X}$ or $X\hat{}$ denote its ℓ-adic completion (for spaces see [13] and for spectra [8, 2.5]). If the homotopy groups of X are finitely generated (and X is simple if X is a space) then there are isomorphisms $\pi_i\hat{X} \cong \mathbb{Z}_\ell \otimes \pi_i X$, where $\mathbb{Z}_\ell$ is the ring of ℓ-adic integers. Let $\mathcal{K}$ denote the topological complex K-theory spectrum and $\hat{\mathcal{K}}^*$ the cohomology theory determined by its completion $\hat{\mathcal{K}}$, so that (by definition) for a spectrum X and integer i, $\hat{\mathcal{K}}^i(X)$ is the abelian group of homotopy classes of maps from X to the i-fold suspension of $\hat{\mathcal{K}}$.

1.1 Topological K-theory of KR_F. We compute $\hat{\mathcal{K}}^*(KR_F)$, the ℓ-adic complex K-theory of KR_F, as a module over the ring of operations in $\hat{\mathcal{K}}^*$. What emerges is a

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startling formula for $\hat{\mathcal{K}}^*(KR_F)$ in terms of classical number theoretic invariants of the ring R_F (1.7).

Associated to the cohomology theory $\hat{\mathcal{K}}^*$ is a *Bousfield localization functor* $\hat{L}$ (see §4). This is a functor on the category of spectra that assigns to each X a spectrum $\hat{L}(X)$ which, in a certain natural sense [8], captures the fragment of X visible to $\hat{\mathcal{K}}^*$.

1.2 $\hat{\mathcal{K}}$-localization of KR_F. Starting from the cohomology calculation 1.1, we give an explicit determination of $\hat{L}(KR_F)$.

Explanation of the results. The reader may wonder why, aside from the alphabetical coincidence, we would choose to apply functors associated with *topological* K-theory (e.g., $\hat{\mathcal{K}}^*$, $\hat{L}$) to an *algebraic* K-theory spectrum. One reason is that we come up with an unexpectedly simple and interesting formula. Another is that the Lichtenbaum-Quillen conjecture (1.4), in one of its many interpretations, states that the $\hat{\mathcal{K}}^*$-localization map $KR_F \to \hat{L}(KR_F)$ should be *close to an equivalence*. In other words, when we compute $\hat{L}(KR_F)$ we are, if the Lichtenbaum-Quillen conjecture is true, essentially computing the algebraic K-theory spectrum KR_F itself.

The spectrum $\hat{L}(KR_F)$ is woven out of topological K-theory according to a pattern provided by R in the same general way as Quillen's spectrum "$F\Psi^q$" is woven according to a pattern provided by the Galois theory of a finite field [34]. What comes out is more complicated than $F\Psi^q$, because R is arithmetically more complicated than a finite field. One thing that we do in this paper is to identify exactly how the classical arithmetic invariants of R govern the construction of an analogue (for R) of what Quillen constructed by hand in the finite-field case.

Some further consequences. There are a number of additional results we obtain as consequences of the ones above. If X is a spectrum, let $\Omega^\infty_0 X$ denote the basepoint component of the zero'th space in the corresponding Ω-spectrum.

1.3 Calculation of $\hat{\mathcal{K}}^* \operatorname{BGL}(R_F)$. The localization map $KR_F \to \hat{L}(KR_F)$ induces an isomorphism on $\hat{\mathcal{K}}^*$, and it follows from arguments of Bousfield [9] that the induced map $\Omega^\infty_0(KR_F) \to \Omega^\infty_0 \hat{L}(KR_F)$ of spaces also induces an isomorphism on $\hat{\mathcal{K}}^*$. There is a natural map $\operatorname{BGL}(R_F) \to \Omega^\infty_0(KR_F)$ which induces an isomorphism on mod ℓ cohomology and therefore also on $\hat{\mathcal{K}}^*$. This implies that the composite map

$$\hat{\mathcal{K}}^* \operatorname{BGL}(R_F) \leftarrow \hat{\mathcal{K}}^* \Omega^\infty_0(KR_F) \leftarrow \hat{\mathcal{K}}^* \Omega^\infty_0 \hat{L}(KR_F)$$

is an isomorphism. The groups on the far right here can in principle be calculated from 1.2. In some cases we can carry out this calculation (with the help of [12]) and end up with an explicit formula for $\hat{\mathcal{K}}^* \operatorname{BGL}(R_F)$.

If S is a ring let $\hat{K}S$ denote the ℓ-completion of the spectrum KS.

1.4 Conjectural determination of $\hat{K}R_F$. The Lichtenbaum-Quillen conjecture for $K\mathcal{O}_F$ amounts (1.11) to the conjecture that the localization map

$$\hat{K}R_F \to \hat{L}(\hat{K}R_F) \simeq \hat{L}(KR_F)$$

is an equivalence on 0-connective covers (i.e., gives an isomorphism on homotopy groups in dimensions ≥ 1). In this way, taking the 0-connective cover of the computation in 1.2 gives an explicit conjectural formula for the 0-connective cover of $\hat{K}R_F$. (The only thing lost in passing from $\hat{K}R_F$ to its 0-connective cover is $\pi_0(\hat{K}R_F) = \mathbb{Z}_\ell \otimes K_0R_F$.) In some cases we are able to show that a factor of $\Omega_0^\infty(\hat{L}KR_F)$ related to the Borel classes [7] is a retract of $\Omega_0^\infty(\hat{K}R_F)$; this could be taken as evidence for the Lichtenbaum-Quillen conjecture.

If X is a space or a spectrum, let $\mathrm{H}^*(X)$ denote $\mathrm{H}^*(X;\mathbb{Z}/\ell)$.

1.5 Conjectural calculation of $\mathrm{H}^*\,\mathrm{BGL}(R_F)$. As in 1.3, there is a natural map $\mathrm{BGL}(R_F) \to \Omega_0^\infty KR_F$ which gives an isomorphism $\mathrm{H}^*\,\Omega_0^\infty(KR_F) \cong \mathrm{H}^*\,\mathrm{BGL}(R_F)$. With the help of 1.2 we are able to calculate $\mathrm{H}^*\,\Omega_0^\infty\hat{L}(KR_F)$. Since $\mathrm{H}^*\,\Omega_0^\infty\hat{L}(KR_F)$ is isomorphic to $\mathrm{H}^*\,\Omega_0^\infty(KR_F)$ if the Lichtenbaum-Quillen conjecture is true, this gives an explicit conjectural formula for $\mathrm{H}^*\,\mathrm{BGL}(R_F)$.

1.6 Harmonic localization of KR_F. Let $\mathcal{K}(i)$ denote the i'th Morava K-theory [38] with respect to the prime ℓ. The harmonic approach to stable homotopy theory suggests studying the localizations $L_n(KR_F)$, where L_n is Bousfield localization (§4) with respect to the wedge $\vee_{i=0}^n\mathcal{K}(i)$, $0 \leq n \leq \infty$. The spectrum $L_0(KR_F)$ is just the rationalization of KR_F and is completely determined by its rational homotopy groups, which were computed by Borel [7]. The spectrum $L_1(KR_F)$ lies in a homotopy fibre square

$$\begin{array}{ccc} L_1(KR_F) & \longrightarrow & \hat{L}(KR_F) \\ \downarrow & & \downarrow \\ L_0(KR_F) & \xrightarrow{b} & L_0\hat{L}(KR_F) \end{array}$$

in which the two spaces on the right are known by 1.2. It follows from [31] that $\mathcal{K}(n)_*KR_F$ vanishes for $n \geq 2$, which implies that $L_n(KR_F) = L_1(KR_F)$ for $n \geq 2$. Up to determining the map b in the above fibre square, then, we have completed the harmonic analysis of KR_F.

There are results analogous to 1.1-1.6 if R_F is replaced by an ℓ-adic local field.

As mentioned above, we assume throughout the paper that ℓ is odd. If $\ell = 2$ and $\sqrt{-1} \in F$, the results (except for the homological calculations in §10) hold with at worst notational changes that will be left to the reader. However, for a general F and $\ell = 2$, our methods break down at several points, principally because the ring R_F can have infinite mod 2 étale cohomological dimension. See [16] for a way to deal with this in some particular cases.

More details. To state our basic results (1.1, 1.2) in more detail we need some notation. On the algebraic side, let F_∞ be the cyclotomic extension of F obtained by adjoining all ℓ-power roots of unity. Let Γ'_F denote the Galois group of F_∞ over F and Λ'_F the completed group ring $\mathbb{Z}_\ell[[\Gamma'_F]]$ (see §2). The *Iwasawa module* M_F is the Galois group of the maximal abelian ℓ-extension of F_∞ that is unramified away from ℓ; it is a profinite Λ'_F-module. Since Γ'_F acts faithfully on the multiplicative group μ_{ℓ^∞} of ℓ-power roots of unity, there is a canonical monomorphism $c_F : \Gamma'_F \to \mathrm{Aut}(\mu_{\ell^\infty}) = \mathbb{Z}_\ell^\times$.

On the topological side, let $\Lambda' = \hat{\mathcal{K}}^0\hat{\mathcal{K}}$ be the ring of degree 0 operations in $\hat{\mathcal{K}}^*$ and $\Gamma' \subset (\hat{\mathcal{K}}^0\hat{\mathcal{K}})^\times$ the group of self-equivalences of $\hat{\mathcal{K}}$ which preserve the ring spectrum structure (Γ' is the group under composition of ℓ-adic Adams operations). There is an isomorphism $\Lambda' \cong \mathbb{Z}_\ell[[\Gamma']]$. The action of Γ' on $\pi_2\hat{\mathcal{K}} \cong \mathbb{Z}_\ell$ gives a canonical isomorphism $c : \Gamma' \cong \mathrm{Aut}(\mathbb{Z}_\ell) = \mathbb{Z}_\ell^\times$, and so the composite homomorphism $(c)^{-1}\cdot c_F$ provides canonical embeddings $\Gamma'_F \subset \Gamma'$ and $\Lambda'_F \subset \Lambda'$. Note that the cohomology theory $\hat{\mathcal{K}}^*$ is periodic of period 2.

1.7 Theorem. *For any number field F there are isomorphisms*

$$\hat{\mathcal{K}}^{-1}KR_F \cong \Lambda' \otimes_{\Lambda'_F} M_F$$
$$\hat{\mathcal{K}}^{0}KR_F \cong \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_\ell$$

of modules over the ring Λ' of degree 0 operations in $\hat{\mathcal{K}}^*$. The spectrum $\hat{L}(KR_F)$ is uniquely determined up to homotopy by $\hat{\mathcal{K}}^*(KR_F)$.

In the second formula of 1.7, the ring Λ'_F acts on $\mathbb{Z}_\ell$ by the augmentation map $\mathbb{Z}_\ell[[\Gamma'_F]] \to \mathbb{Z}_\ell$, so that $\hat{\mathcal{K}}^0KR_F$ is actually isomorphic to $\mathbb{Z}_\ell[\Gamma'/\Gamma'_F]$. The last assertion of the theorem follows ultimately from a splitting result (9.7) and the fact that M_F is known to have projective dimension at most one as a Λ'_F-module. It is sometimes possible to give an explicit realization of $\hat{L}(KR_F)$ as the fibre of a map between finite wedges of copies of suspensions of retracts of $\hat{\mathcal{K}}$.

The result analogous to 1.7 for local fields runs as follows. If E is an ℓ-adic local field, i.e., a finite extension of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, define E_∞, Γ'_E, Λ'_E as before, and, also as before, let M_E denote the Galois group over E_∞ of the maximal abelian ℓ-extension of E_∞. Then

1.8 Theorem. *For any ℓ-adic local field E there are isomorphisms*

$$\hat{\mathcal{K}}^{-1}KE \cong \Lambda' \otimes_{\Lambda'_E} M_E$$
$$\hat{\mathcal{K}}^{0}KE \cong \Lambda' \otimes_{\Lambda'_E} \mathbb{Z}_\ell$$

of modules over the ring Λ' of degree 0 operations in $\hat{\mathcal{K}}^*$. The spectrum $\hat{L}(KE)$ is uniquely determined up to homotopy by $\hat{\mathcal{K}}^*(KE)$.

Remark. Theorem 1.8 can be made very concrete, in the sense that the "Iwasawa module" M_E (and hence the localization $\hat{L}(KE)$) can be described entirely in terms of the degrees of certain field extensions (13.3). There is no simple general expression like this for the Iwasawa module M_F of a number field F.

1.9 Motivation. We will now describe a point of view from which the formulas in 1.7 are almost obvious, although making this point of view precise would involve technicalities we do not want to treat here. If X is a space let X_+ denote the unreduced suspension spectrum of X. If U and V are spectra, let $\mathrm{Map}(U,V)$ denote the spectrum of maps between them, so that in particular

$$\hat{\mathcal{K}}^i(U) = \pi_{-i}\,\mathrm{Map}(U,\hat{\mathcal{K}})\,.$$

If X is a finite CW-complex, a straightforward argument with Spanier-Whitehead duality shows that there are natural isomorphisms

$$\hat{\mathcal{K}}^i \operatorname{Map}(X_+, \hat{\mathcal{K}}) \cong \pi_{-i}(X_+ \wedge \operatorname{Map}(\hat{\mathcal{K}}, \hat{\mathcal{K}}))\,.$$

Note here that $\operatorname{Map}(\hat{\mathcal{K}}, \hat{\mathcal{K}})$ is a $\hat{\mathcal{K}}$-module spectrum whose even homotopy groups are isomorphic to Λ' and whose odd homotopy groups are zero. More generally, suppose that X is connected, let $\theta : \pi_1 X \to \Gamma' \subset (\Lambda')^\times$ be a homomorphism with image G, let X^θ be the cover of X corresponding to the kernel of θ, and define the θ-twisted K-theory $\hat{\mathcal{K}}^*_\theta(X)$ of X by

$$\hat{\mathcal{K}}^i_\theta(X) = \pi_{-i} \operatorname{Map}_G(X^\theta_+, \hat{\mathcal{K}})\,.$$

Here the mapping spectrum on the right is an equivariant one; in order to define it easily the action of Γ' on $\hat{\mathcal{K}}$ must be given rigidly and not just up to homotopy. An equivariant duality argument gives the formula

$$\hat{\mathcal{K}}^i \operatorname{Map}_G(X^\theta_+, \hat{\mathcal{K}}) \cong \pi_{-i}(X^\theta_+ \wedge_G \operatorname{Map}(\hat{\mathcal{K}}, \hat{\mathcal{K}}))\,.$$

Now let F be a number field and X the étale homotopy type $X_{\text{ét}}(R_F)$ of R_F [19]; by definition X is a connected pro-space whose fundamental group is the Galois group over F of the maximal extension of F which is unramified away from ℓ. It turns out (3.1) that the pro-space X has the mod ℓ cohomological properties of a 2-dimensional CW-complex. The action of $\pi_1 X$ on μ_{ℓ^∞} gives a homomorphism

$$\theta : \pi_1 X \twoheadrightarrow \Gamma'_F \subset \Gamma'$$

and it is more or less clear from Thomason's étale descent spectral sequence [43] that there should be natural isomorphisms

$$\pi_i \hat{L}(KR_F) \cong \hat{\mathcal{K}}^i_\theta(X) \cong \pi_i \operatorname{Map}_{\Gamma'_F}(X^\theta_+, \hat{\mathcal{K}})\,.$$

As above, then

$$\begin{aligned} \hat{\mathcal{K}}^i \hat{L}(KR_F) &\cong \hat{\mathcal{K}}^i \operatorname{Map}_{\Gamma'_F}(X^\theta_+, \hat{\mathcal{K}}) \\ &\cong \pi_{-i}(X^\theta_+ \wedge_{\Gamma'_F} \operatorname{Map}(\hat{\mathcal{K}}, \hat{\mathcal{K}}))\,. \end{aligned}$$

To calculate this it is necessary to know something about X^θ, a pro-space which is $X_{\text{ét}}(R_{F_\infty})$ for the ring R_{F_∞} of algebraic ℓ-integers in F_∞. To begin with, $\mathrm{H}_i(X^\theta; \mathbb{Z}_\ell) = 0$ for $i \geq 2$, essentially because X has mod ℓ (co-)homological dimension at most 2 and X^θ is tantamount to an infinite cyclic cover of X. The pro-space X^θ is connected and so $\mathrm{H}_0(X^\theta, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$. Finally, almost by definition $\mathrm{H}_1(X^\theta, \mathbb{Z}_\ell) \cong M_F$, since $X^\theta = X_{\text{ét}}(R_{F_\infty})$ and M_F is the Galois group of the maximal abelian ℓ-extension of F_∞ which is unramified away from ℓ. It follows that X^θ_+ is an unreduced Moore spectrum of type $(M_F, 1)$. With a little technical good will, the calculation

$$\pi_i(X^\theta_+ \wedge_{\Gamma'_F} \operatorname{Map}(\hat{\mathcal{K}}, \hat{\mathcal{K}})) \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} M_F & i = 1 \\ \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_\ell & i = 0 \end{cases}$$

is immediate.

1.10 Method of Proof. Our technique for proving 1.7 is to start with Thomason's spectral sequence for $\pi_*\hat{L}(KR_F)$ and work backwards to $\hat{\mathcal{K}}^*(KR_F)$. What makes this unlikely project successful is the fact (5.2) that some spectra which arise are module spectra over $\hat{\mathcal{K}}$. For such module spectra X there is a very close relationship between π_*X and $\hat{\mathcal{K}}^*X$ (see the proof of 6.11). We conjecture (8.6) that related module spectrum structures exist on unlocalized algebraic K-theory spectra.

The tools we use involve basic étale cohomology theory (§3), Thomason's spectral sequence (6.1), and a theorem of Iwasawa (8.1). In working out the consequences of 1.7 we develop additional parts of Iwasawa theory (8.10, 12.1). This paper could not have been written without the benefit of Bousfield's extensive work on topological K-theory.

1.11 Relationship to étale K-theory. Let $K^{\text{ét}}R_F$ denote the étale K-theory spectrum of R_F constructed in [15]. By comparing the spectral sequence of [15, 5.2] with Thomason's spectral sequence, it is possible to show that the map $\hat{K}R_F \to \hat{L}(KR_F)$ is equivalent to the map $\hat{K}R_F \to K^{\text{ét}}R_F$ of [15], at least after passing to 0-connective covers [43, 4.11]. Thus the present paper can be interpreted as giving a calculation both of $\hat{\mathcal{K}}^*(K^{\text{ét}}R_F)$ (cf. 4.9) and of the homotopy type of the 0-connective cover of $K^{\text{ét}}R_F$. This generalizes results of [16] and [17] which describe the space $\Omega_0^\infty K^{\text{ét}}R_F$ for some special fields F.

The remark in [15, 8.8] is the basis for the statement that the Lichtenbaum-Quillen conjecture is true for R_F if and only if the map $\hat{K}R_F \to \hat{L}(\hat{K}R_F)$ is an equivalence on 0-connective covers.

1.12 Organization of the paper. The first four sections are preparatory. Section 2 establishes number-theoretic notation, and §3 contains a discussion of étale cohomology. Section 4 sketches Bousfield's theory of $\hat{\mathcal{K}}^*$-localization and derives some properties of $\hat{\mathcal{K}}^*$-local spectra. In §5 there is a construction of the key module spectrum structures mentioned in 1.10. Section 6 has a proof of Theorem 1.7 and a description of the functoriality properties of the isomorphisms this theorem provides. In §7 we derive some homological properties of modules over the Iwasawa algebra, and then use these properties in §8 and §9 to study the wedge summand $\hat{L}(K^{\text{red}}R_F)$ of $\hat{L}(KR_F)$ (see 9.7) and to show that in some situations wedge factors of $\hat{L}(K^{\text{red}}R_F)$ give cartesian factors of $\mathrm{BGL}(R)\hat{}$. Our conjectural formula for $\mathrm{H}^*\,\mathrm{BGL}(R_F)$ is in §10 and our (non-conjectural) calculations of $\hat{\mathcal{K}}^*\,\mathrm{BGL}(R_F)$ are in §11. Finally, §12 works out explicit examples of the theory for particular fields F, and §13 sketches the extension to ℓ-adic local fields.

1.13 Notation and terminology. In general we will used $\mathcal{K}$ for topological variants of K-theory and K for algebraic K-theory. Our algebraic notation is described in §2; note also the definition of reduced K-theory spectra $K^{\text{red}}R_F$ and $\hat{K}^{\text{red}}R_F$ in 2.1. We assume that all spectra in this paper have been localized at ℓ (4.2). Except in special cases all maps between spectra are in the *homotopy category*, so that diagrams of spectra commute up to homotopy, groups act on spectra up to homotopy, etc. If X is a space, then X_+ is the unreduced suspension spectrum of X. If G is a (topological) group, then $\hat{\mathrm{B}}G$ is the ℓ-completion of $\mathrm{B}G$ and $\hat{\mathrm{B}}G_+$ the

ℓ-completion of the unreduced suspension spectrum of BG. For a spectrum X, P^kX is the k-connective cover of X, so that $\pi_i P^kX$ is zero for $i \le k$ and there is a map $P^kX \to X$ inducing isomorphisms $\pi_i P^kX \cong \pi_i X$ for $i > k$. For an abelian group N and integer k, $\mathcal{M}(N,k)$ is a Moore spectrum of type (N,k), so that $\mathrm{H}_i(\mathcal{M}(N,k);\mathbb{Z})$ is N if $i = k$ and 0 otherwise. The symbol $\mathcal{M}_{\ell^k}$ denotes $\mathcal{M}(\mathbb{Z}/\ell^k, 0)$. Topological K-theory Moore spectra $\mathcal{M}_{\mathcal{K}}(N,k)$ are defined for suitable N in 4.17. Unless otherwise indicated, homology and cohomology groups have coefficients $\mathbb{Z}/\ell$.

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§2. Number fields and rings of integers

In this section we will describe the notation we use for various objects associated to a number field.

Let F be a fixed algebraic number field, in other words, a finite extension field of $\mathbb{Q}$. For concreteness, we will assume that F is contained in the field $\mathbb{C}$ of complex numbers. Recall from §1 that ℓ is a fixed odd prime. Let $F_0 = F(\mu_\ell)$, where in general $\mu_{\ell^\nu} \subset \mathbb{C}^\times$ is the multiplicative group of ℓ^ν'th roots of unity, let $d = d_F$ be the degree of F_0 over F, and let $a = a_F$ denote the maximal value of ν such that $\mu_{\ell^\nu} \subset F_0$. If $F_\infty = F_0(\mu_{\ell^\infty})$, then $F_\infty = \bigcup_{n\ge 0} F_n$, where $F_n = F_0(\mu_{\ell^{a+n}})$ and F_n is a cyclic Galois extension of degree ℓ^n over F_0. For notational purposes it will sometimes be convenient to refer to F as F_{-1}. Let Γ'_F denote the Galois group $\mathrm{Gal}(F_\infty/F)$, Γ_F the group $\mathrm{Gal}(F_\infty/F_0)$, and Δ_F the quotient group $\Gamma'_F/\Gamma_F = \mathrm{Gal}(F_0/F)$. We let Λ_F denote the completed group ring $\mathbb{Z}_\ell[[\Gamma_F]]$; this is defined to be the inverse limit over n of the rings $\mathbb{Z}_\ell[\Gamma_n]$, where $\Gamma_n = \mathrm{Gal}(F_n/F_0)$. The symbol Λ'_F will denote the slightly larger but similarly defined ring $\mathbb{Z}_\ell[[\Gamma'_F]]$. The *Iwasawa module* $M = M_F$ is the Galois group of the maximal abelian ℓ-extension of F_∞ which is unramified away from ℓ [44, §13.5]; there is a natural action of Γ'_F on M which makes M into a profinite module over Λ'_F [25, p. 145].

The action of Γ'_F on μ_{ℓ^∞} defines a canonical embedding $c_F : \Gamma'_F \to \mathrm{Aut}(\mu_{\ell^\infty}) = \mathbb{Z}_\ell^\times$ which restricts to an isomorphism between Γ_F and the multiplicative group of ℓ-adic units congruent to 1 mod ℓ^a. It follows that Γ_F is abstractly isomorphic to the topologically cyclic group $\mathbb{Z}_\ell$. The quotient group Δ_F of Γ'_F acts faithfully on μ_ℓ and so has order dividing the order of $\mathrm{Aut}(\mu_\ell)$, which is $(\ell - 1)$. In particular $(|\Delta_F|, \ell) = 1$, so the extension

$$\{1\} \to \Gamma_F \to \Gamma'_F \to \Delta_F \to \{1\}$$

splits uniquely, and c_F restricts to a canonical embedding $\omega : \Delta_F \subset \Gamma'_F \to \mathbb{Z}_\ell^\times$, called the "Teichmuller character". The group Γ'_F is also topologically cyclic, and γ'_F will denote a chosen generator. Let $\mathbb{Z}_\ell(m)$ be the abelian group $\mathbb{Z}_\ell$ with the Γ'_F-action for which $g \in \Gamma'_F$ acts by multiplication by $c_F(g)^m$. If N is a $\mathbb{Z}_\ell$-module with an action of Γ'_F, the *m-fold Tate twist* $N(m)$ is the tensor product $\mathbb{Z}_\ell(m) \otimes_{\mathbb{Z}_\ell} N$ with the diagonal Γ'_F-action.

Let R_n $(-1 \le n < \infty)$ denote the *ring of algebraic ℓ-integers* in F_n, that is, $R_n = \mathcal{O}_{F_n}[1/\ell]$, where $\mathcal{O}_{F_n}$ is the ring of algebraic integers in F_n. The ℓ-torsion subgroup of the class group of $\mathcal{O}_{F_n}$ is denoted A_n, the ℓ-completion of the unit group of $\mathcal{O}_{F_n}$ is denote E_n, and the corresponding groups for R_n are denoted A'_n and E'_n respectively. Let S_n denote the set of primes over ℓ in R_n and $s_n = |S_n|$ its cardinality. The primes of R_{-1} over ℓ are eventually totally ramified in the cyclotomic tower (cf. [44, 13.3]), so s_n eventually stabilizes to a finite value $s_\infty = s_n$, $n \gg 0$. Define B_n to be $\mathrm{Hom}(\mathbb{Z}/\ell^\infty, \mathrm{Br}(R_n))$, where $\mathrm{Br}(R_n)$ is the Brauer group; as a module over Γ'_F, B_n is just the reduced permutation module $B_n = \ker(\epsilon : \mathbb{Z}_\ell[S_n] \to \mathbb{Z}_\ell)$, where ϵ is the natural augmentation. The number $r_1(F_n)$ is the number of embeddings of F in $\mathbb{R}$, and $r_2(F_n))$ the number of complex conjugate pairs of non-real embeddings of F_n in $\mathbb{C}$ (note that $r_1(F_n) = 0$ unless $n = -1$). Finally, $A_\infty = \lim_n A_n$, $A'_\infty = \lim_n A'_n$, $E'_\infty = \lim_n E'_n$ and $B_\infty = \lim_n B_n$, where the inverse limits are taken with respect to the norm or transfer maps. These inverse limits are all profinite Λ'_F-modules, and in the case of B_∞ the inverse system stabilizes, so that $B_\infty = B_n$, $n \gg 0$.

Symbols without subscripts (A', E', etc.) stand for the corresponding objects associated to $F = F_{-1}$; in particular, R is the ring of algebraic ℓ-integers in F.

Fix once and for all a prime $\mathcal{P}$ of R such that $\mathcal{P}$ remains prime in R_n for all $n \ge 0$. This property of $\mathcal{P}$ is equivalent to the condition that the number $q = |R/\mathcal{P}|$ be a topological generator of $c_F(\Gamma'_F) \subset \mathbb{Z}_\ell^\times$. An infinite number of such primes exist by the Cebotarev density theorem. Let $\mathbb{F} = R/\mathcal{P}$, $\mathbb{F}_n = R_n/\mathcal{P}$ and $\mathbb{F}_\infty = R_\infty/\mathcal{P}$, where $R_\infty = \cup_n R_n$ is the ring of algebraic ℓ-integers in F_∞. The choice of $\mathcal{P}$ implies that $\mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F})$ is isomorphic to Γ'_F. It also implies the natural maps $K_i R_n \to K_i \mathbb{F}_n$ are split epimorphisms for all $i \ge 0$ and $n \ge -1$ (see [22], [18], or the proof of 5.8).

2.1 Definition. The *reduced K-theory spectrum* of R_n, denoted $K^{\mathrm{red}} R_n$, is the homotopy fibre of the map $KR_n \to K\mathbb{F}_n$ induced by the ring homomorphism $R_n \to \mathbb{F}_n$. The spectrum $\hat{K}^{\mathrm{red}} R_n$ is the ℓ-completion of $K^{\mathrm{red}} R_n$. The *reduced unit group* $E_n(\mathrm{red})$ is the kernel of the reduction map $E_n \to (\mathbb{F}_n^\times)\hat{}$, and $E'_n(\mathrm{red})$ is the kernel of the reduction map $E'_n \to (\mathbb{F}_n^\times)\hat{}$. The Λ'_F-modules $E_\infty(\mathrm{red})$ and $E'_\infty(\mathrm{red})$ are defined respectively as $\lim_n E_n(\mathrm{red})$ and $\lim_n E'_n(\mathrm{red})$, where the limits are taken with respect to norm maps.

Note that there are isomorphisms $\pi_1 \hat{K}^{\mathrm{red}} \mathcal{O}_{F_n} \cong E_n(\mathrm{red})$ and $\pi_1 \hat{K}^{\mathrm{red}} R_n \cong E'_n(\mathrm{red})$. By 9.7 there is a splitting $\hat{L}(KR_n) \simeq \hat{L}(K\mathbb{F}_n) \vee \hat{L}(K^{\mathrm{red}} R_n)$, and we conjecture (8.6) that there is such a splitting of KR_n.

§3. Étale cohomology

In this section we recall some facts from étale cohomology theory and draw some consequences (3.5, 3.8) which we will need later on. Étale cohomology theory plays a role in the paper because of Thomason's theorem (6.1), which gives a relationship between $\pi_* \hat{L}(KR)$ and étale cohomology groups.

The part of étale cohomology theory which we will use has a strong topological flavor. Associated to any commutative noetherian ring S is an étale topological type $X_{\text{ét}}(S)$ [19, 4.4], which is a *pro-space*, i.e., an inverse system in the category of spaces. The fundamental group $\pi_1 X_{\text{ét}}(S)$ is ordinarily a profinite group [19, 7.3] (we are suppressing here the need to choose a basepoint). If S is a field then $\pi_1 X_{\text{ét}}(S)$ is the Galois group over S of the separable closure of S, while if S is the ring of ℓ-integers in a number field E, then $\pi_1 X_{\text{ét}}(S)$ is the Galois group over E of the maximal extension of E which is unramified away from ℓ [19, 5.6]. The assignment $S \mapsto X_{\text{ét}}(S)$ is contravariant and can be thought of as a topological enrichment of Galois theory, in that it associates to S a pro-space whose fundamental group is typically a Galois group.

If $S = R$ or $S = \mathbb{F}$ (or more generally if $1/\ell \in S$) then adjoining an ℓ-primary root of unity gives an étale extension of S [14, p. 21]. This implies that there is a natural map $\pi_1 X_{\text{ét}}(S) \to \operatorname{Aut}(\mu_{\ell^\infty}) \cong \Gamma'$, so that a Γ'-module such as $\mathbb{Z}/\ell^k(m) = \mathbb{Z}/\ell^k \otimes \mathbb{Z}_\ell(m)$ gives a local coefficient system [15, §5] [19, §5] on $X_{\text{ét}}(S)$. The local coefficient cohomology groups $\mathrm{H}^i(X_{\text{ét}}(S); \mathbb{Z}/\ell^k(m))$ are the basic objects we are interested in; these can be identified with the *étale cohomology groups* $\mathrm{H}^i_{\text{ét}}(S; \mathbb{Z}/\ell^k(m))$ of S [19, 5.9, 2.4]. For a field S, étale cohomology agrees with Galois cohomology [14, p. 24]. We need three main properties of these groups.

3.1 Theorem. [5] [39, 3.3, §5] [40, III.1] *Suppose that N is a finite ℓ-torsion Γ'-module. If S is a local field or the ring of ℓ-integers in a number field, the group $\mathrm{H}^i_{\text{ét}}(S; N)$ vanishes for $i \geq 3$ and is finite otherwise. If S is a finite field $\mathbb{F}$, the group $\mathrm{H}^i_{\text{ét}}(S; N)$ vanishes for $i \geq 2$ and is finite otherwise.*

We will use the following theorem only for field extensions and for the ring extensions R_m/R_n, $m > n$.

3.2 Theorem. *Suppose that $1/\ell \in S$, and that $\tilde{S}/S$ is a finite Galois extension of rings with group $G = \operatorname{Gal}(\tilde{S}/S)$. Then there is a natural action of G on the groups $\mathrm{H}^*_{\text{ét}}(\tilde{S}, \mathbb{Z}/\ell^k(m))$ and a first quadrant spectral sequence of cohomological type*

$$E_2^{i,j} = \mathrm{H}^i(G; \mathrm{H}^j_{\text{ét}}(\tilde{S}; \mathbb{Z}/\ell^k(m)) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(S, \mathbb{Z}/\ell^k(m))\,.$$

Proof. This amounts to the local coefficient Serre spectral sequence of the fibration

$$X_{\text{ét}}(\tilde{S}) \to X_{\text{ét}}(S) \to \mathrm{B}G$$

associated to the regular covering $X_{\text{ét}}(\tilde{S}) \to X_{\text{ét}}(S)$ with group G [19, 5.3, 5.6]. See [39, I 2.6] for the special case of a field extension and [3, VIII, 8.5] in general. □

If $1/\ell \in S$, we let $\mathrm{H}^*_{\text{ét}}(S; \mathbb{Z}_\ell(m))$ denote the *continuous étale cohomology* of S with coefficients in $\mathbb{Z}_\ell(m)$ constructed by Jannsen [24].

3.3 Remark. For any S and $i \geq 0$, the cohomology maps induced by the natural surjections $\mathbb{Z}/\ell^{k+1}(m) \to \mathbb{Z}/\ell^k(m)$ give an inverse system $\{\mathrm{H}^i_{\text{ét}}(S; \mathbb{Z}/\ell^k(m))\}_k$. The finiteness statements in 3.1 imply that if S is a finite field or the ring of ℓ-integers in a global field, then for each $i \geq 0$ this inverse system is Mittag-Leffler. By [24], then, for each $i \geq 0$ the group $\mathrm{H}^i_{\text{ét}}(S; \mathbb{Z}_\ell(m))$ is isomorphic to $\lim_k \mathrm{H}^i_{\text{ét}}(S; \mathbb{Z}/\ell^k(m))$.

The continuous étale cohomology groups have usual exactness properties. For instance, there is a long exact sequence involving ordinary and continuous étale cohomology groups corresponding to the short exact coefficient sequence

$$0 \to \mathbb{Z}_\ell(m) \xrightarrow{\ell^k} \mathbb{Z}_\ell(m) \to \mathbb{Z}/\ell^k(m) \to 0\,.$$

3.4 Theorem. [40, II.1.4] [41] *Suppose that* $1/\ell \in S$ *and consider the group* $H^i_m = \mathrm{H}^i_{\text{ét}}(S, \mathbb{Z}_\ell(m))$*. If* S *is a local field, then* H^i_m *is zero for* $i \geq 3$ *and finite for* $i = 2$ *unless* $m = 1$*. If* S *is the ring of* ℓ*-integers in a number field, the group* H^i_m *is zero for* $i \geq 3$ *and finite for* $i = 2$ *if* $m \geq 2$*. If* S *is a finite field,* H^i_m *is zero for* $i \geq 2$ *and finite for* $i = 1$ *unless* $m = 0$*. In all three cases* H^i_m *is zero for* $i = 0$ *unless* $m = 0$*.*

Remark. The difficult part of 3.4 for a ring S of ℓ-integers in a number field is the statement that $\mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(m))$ is finite for $m \geq 2$. To prove this using the étale K-theory of [15], note that by [15] there is a surjective map

$$\pi_{2m-2}\hat{K}S \to K^{\text{ét}}_{2m-2}S \cong \mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(m))$$

while by theorems of Quillen and Borel the group $\pi_{2m-2}\hat{K}S$ is finite.

If $1/\ell \in S$, define

$$\mathrm{H}^*_{\text{ét}}(S, \mathbb{Z}/\ell^\infty(m)) = \mathrm{colim}_k\, \mathrm{H}^*_{\text{ét}}(S, \mathbb{Z}/\ell^k(m))$$

where the colimit is taken with respect to the cohomology maps induced by the usual inclusions $\mathbb{Z}/\ell^k(m) \to \mathbb{Z}/\ell^{k+1}(m)$. By [3, VII, 5.7] there are isomorphisms

$$\mathrm{H}^*_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^k(m)) \cong \mathrm{colim}_n\, \mathrm{H}^*_{\text{ét}}(R_n; \mathbb{Z}/\ell^k(m))$$

and so taking another colimit give isomorphisms

$$\mathrm{H}^*_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m)) \cong \mathrm{colim}_n\, \mathrm{H}^*_{\text{ét}}(R_n; \mathbb{Z}/\ell^\infty(m))\,.$$

Let $A^\#$ denote the Pontriagin dual $\mathrm{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ of the profinite abelian group A.

3.5 Lemma. *The group* $\mathrm{H}^1_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m))$ *is naturally isomorphic as a module over* Γ'_F *to the twisted Pontriagin dual* $M^\#(m)$ *of the Iwasawa module* M*.*

Proof. Note first that for any m the local coefficient system $\mathbb{Z}/\ell^\infty(m)$ on $X_{\text{ét}}(R_\infty)$ is trivial; the only role played by m in the lemma is to determine the action of Γ'_F on the cohomology group involved. Let G be the Galois group over F_∞ of the maximal extension of F_∞ which is unramified away from ℓ. The pro-space version of the universal coefficient theorem in dimension 1 gives isomorphisms

$$\mathrm{H}^1_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m)) \cong \mathrm{Hom}_{\text{cont}}(\pi_1 X_{\text{ét}}(R_\infty), \mathbb{Z}/\ell^\infty) \cong \mathrm{Hom}_{\text{cont}}(G, \mathbb{Z}/\ell^\infty(m))\,.$$

However, the group on the right here is isomorphic to $M^\#$, because by definition M is the maximal abelian ℓ-profinite quotient of G. Under this isomorphism, the action of $g \in \Gamma'_F$ on a homomorphism $f : M \to \mathbb{Z}/\ell^\infty(m)$ sends f to f', where $f'(x) = c_F(g)^m f(g^{-1}x)$. This gives the desired result. □

3.6 Lemma. *If S is the ring of ℓ-integers in a number field and $m \geq 2$, then the group $\mathrm{H}^i_{\text{ét}}(S; \mathbb{Z}/\ell^\infty(m))$ vanishes for $i \geq 2$. If S is a finite field of characteristic prime to ℓ and $m \neq 0$, then the group $\mathrm{H}^i_{\text{ét}}(S; \mathbb{Z}/\ell^\infty(m))$ vanishes for $i \geq 1$.*

3.7 Remark. By taking colimits it is possible to prove parallel vanishing theorems for $\mathrm{H}^i_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m))$ and $\mathrm{H}^i_{\text{ét}}(\mathbb{F}_\infty; \mathbb{Z}/\ell^\infty(m))$. In fact, for these rings the question of whether the cohomology groups vanish is independent of m (cf. proof of 3.5) and so we can conclude that the groups vanish (uniformly in m) for $i \geq 2$ in the case of R_∞ and for $i \geq 1$ in the case of $\mathbb{F}_\infty$.

Proof of 3.6. We will treat the case in which S is the ring of ℓ-integers in a number field; the other case is similar. By 3.4 we have only to check the vanishing for $i = 2$. Again by 3.4 the group $\mathrm{H}^3_{\text{ét}}(S; \mathbb{Z}_\ell(m))$ vanishes, and so by the long exact cohomology sequence mentioned in 3.3 there are isomorphisms $\mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}/\ell^k(m)) \cong \mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(m)) \otimes \mathbb{Z}/\ell^k$ for any k. Taking a colimit over k shows that $\mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}/\ell^\infty(m))$ is isomorphic to $\mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(m)) \otimes \mathbb{Z}/\ell^\infty$, which vanishes because $\mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(m))$ is finite (3.4). □

We will make use of the following theorem, which is certainly not new. Recall that γ'_F is a topological generator of the profinite topologically cyclic group Γ'_F.

3.8 Theorem. *For each $m \geq 2$ if $i = 1$ and for $m \geq 1$ if $i = 0$ the natural sequences*

$$0 \to \mathrm{H}^i_{\text{ét}}(R; \mathbb{Z}/\ell^\infty(m)) \to \mathrm{H}^i_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m)) \xrightarrow{\mathrm{id} - \gamma'_F} \mathrm{H}^i_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m)) \to 0$$

$$0 \to \mathrm{H}^i_{\text{ét}}(\mathbb{F}; \mathbb{Z}/\ell^\infty(m)) \to \mathrm{H}^i_{\text{ét}}(\mathbb{F}_\infty; \mathbb{Z}/\ell^\infty(m)) \xrightarrow{\mathrm{id} - \gamma'_F} \mathrm{H}^i_{\text{ét}}(\mathbb{F}_\infty; \mathbb{Z}/\ell^\infty(m)) \to 0$$

are exact.

Remark. By 3.6, the groups in 3.8 vanish if $i \geq 2$.

The following lemma is a special case of the standard formula for the continuous cohomology of a topologically cyclic profinite group with coefficients in a discrete module. The proof comes down to a routine calculation with the cohomology of cyclic groups.

3.9 Lemma. *Suppose that*

$$C_0 \to C_1 \to C_2 \to \cdots$$

is a direct system of ℓ-torsion abelian groups, where C_n is a module over Γ'_F/Γ'_{F_n} and $C_n \to C_{n+1}$ is equivariant with respect to $\Gamma'_F/\Gamma'_{F_{n+1}}$. Let $C = \operatorname{colim}_n C_n$, so that Γ'_F acts on C. Then there are natural isomorphisms

$$\operatorname{colim}_n \mathrm{H}^i(\Gamma'_F/\Gamma'_{F_n}; C_n) \cong \begin{cases} \ker(\mathrm{id} - \gamma'_F) : C \to C & i = 0 \\ \operatorname{coker}(\mathrm{id} - \gamma'_F) : C \to C & i = 1 \\ 0 & i \geq 2 \end{cases}$$

Proof of 3.8. We will only treat the case of the number ring R; the case of the finite field $\mathbb{F}$ is simpler. According to 3.2, for each pair (n, k) of positive integers there is a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\Gamma'_F/\Gamma'_{F_n}; \mathrm{H}^j_{\text{ét}}(R_n; \mathbb{Z}/\ell^k(m))) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(R; \mathbb{Z}/\ell^k(m)) .$$

Take a directed colimit of these spectral sequences (over n and k) to get a new spectral sequence

$$E_2^{i,j} = \operatorname{colim}_n \mathrm{H}^i(\Gamma'_F/\Gamma'_{F_n}; \mathrm{H}^j_{\text{ét}}(R_n; \mathbb{Z}/\ell^\infty(m))) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}(R; \mathbb{Z}/\ell^\infty(m)) .$$

Lemma 3.9 shows that the E_2 term of this spectral sequence vanishes unless $\{i, j\} \subset \{0, 1\}$, and moreover that the statement we are trying to prove is equivalent to the statement that $E_2^{1,0} = 0 = E_2^{1,1}$. Again by 3.9, $E_2^{1,0}$ is isomorphic to the cokernel of $(\mathrm{id} - \gamma'_F)$ acting on $\mathrm{H}^0_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m)) \cong \mathbb{Z}/\ell^\infty(m)$. This cokernel is zero if $m \neq 0$, since any nontrivial endomorphism of $\mathbb{Z}/\ell^\infty$ has a trivial cokernel. Given the sparseness of the E_2-term, the spectral sequence itself computes that $E_2^{1,1}$ is isomorphic to $\mathrm{H}^2_{\text{ét}}(R; \mathbb{Z}/\ell^\infty(m))$, and so vanishes by 3.6. □

We will also need the following calculation. The notation is from §2.

3.10 Proposition. *For each n there is a natural isomorphism $\mathrm{H}^1_{\text{ét}}(R_n; \mathbb{Z}_\ell(1)) \cong E'_n$, as well as a natural short exact sequence*

$$0 \to A'_n \to \mathrm{H}^2_{\text{ét}}(R_n; \mathbb{Z}_\ell(1)) \to B_n \to 0 .$$

Proof. (see [29]) The multiplicative group $\mathbb{G}_\mathrm{m}$ is a sheaf in the étale topology on R_n. Since $1/\ell \in R_n$, there are short exact sequences

$$0 \to \mathbb{Z}/\ell^k(1) \to \mathbb{G}_\mathrm{m} \xrightarrow{\ell^k} \mathbb{G}_\mathrm{m} \to 0$$

of sheaves. The proposition results from examining the associated long exact cohomology sequences, using the standard formulas

$$\mathrm{H}^i_{\text{ét}}(R_n; \mathbb{G}_\mathrm{m}) = \begin{cases} R_n^\times & i = 0 \\ \mathrm{Cl}(R_n) & i = 1 \\ \mathrm{Br}(R_n) & i = 2 \end{cases}$$

and passing to an inverse limit (3.3). Here $\mathrm{Cl}(R_n)$ is the ideal class group of R_n. By class field theory, the Brauer group $\mathrm{Br}(R_n)$ is isomorphic to the kernel of the augmentation map $\oplus_{\mathcal{P} \in S_n} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$.

This argument yields a short exact sequence like the one claimed, but with A'_n replaced by the ℓ-adic completion of $\mathrm{Cl}(R_n)$. However, because the class group $\mathrm{Cl}(R_n)$ is finite, its ℓ-adic completion is isomorphic to A'_n. □

§4. Stable homotopy theory

In this section we recall some facts from stable homotopy theory that are needed for the rest of the paper. We will concentrate on the theory of Bousfield localization with respect to a homology theory, and in particular with respect to mod ℓ K-homology.

Suppose that E is a homology theory, or equivalently, that E is a spectrum with associated homology theory $E_*X = \pi_*(E \wedge X)$. A map $f : X \to Y$ of spectra is said to be an *E_*-equivalence* if the induced map $E_*f : E_*X \to E_*Y$ is an isomorphism. A spectrum Z is said to be *E_*-local* if any E_*-equivalence $f : X \to Y$ induces a bijection $[Y, Z] \to [X, Z]$. (Here $[-,-]$ denotes the group of homotopy classes of maps between two spectra.) Bousfield has proved the following fundamental theorem.

4.1 Theorem. [8] *Let E be a homology theory. Then there exists an E_*-localization functor L_E, which assigns to a spectrum X an E_*-local spectrum $L_E(X)$ together with an E_*-equivalence $X \to L_E(X)$.*

4.2 Remark. If E is the homology theory given by the mod ℓ Moore spectrum $\mathcal{M}_\ell = \mathcal{M}(\mathbb{Z}/\ell, 0)$, in other words, if E is mod ℓ stable homotopy, then L_E is the ℓ-adic completion functor $X \mapsto \hat{X}$. If E is the homology theory given by $\mathcal{M}(\mathbb{Z}_{(\ell)}, 0)$, where $\mathbb{Z}_{(\ell)}$ is the localization of the integers at ℓ, then L_E is the usual arithmetic localization functor with $\pi_i L_E(X) = \mathbb{Z}_{(\ell)} \otimes \pi_i X$.

Note that a spectrum X is E_*-local if and only if the map $X \to L_E X$ is an equivalence.

4.3 Remark. It is easy to see that for any homology theory E and any spectra X and Y there is a natural homotopy equivalence $L_E(L_E X \wedge L_E Y) \simeq L_E(X \wedge Y)$. It follows that if X is a ring spectrum then $L_E(X)$ is a ring spectrum and the map $X \to L_E(X)$ is a map of ring spectra. Similarly, if Y is a module spectrum over X then $L_E(Y)$ is a module spectrum over $L_E(X)$. The localization functor L_E preserves fibration sequences and behaves well with respect to colimits in the sense that if X_i is a filtered direct system of spectra, then there is a natural homotopy equivalence $L_E(\operatorname{colim}_i L_E(X_i)) \cong L_E(\operatorname{colim}_i X_i)$.

We will be particularly interested in localizations associated to topological complex K-theory. Let $\mathcal{K}$ denote the ordinary periodic complex K-theory ring spectrum and $\hat{\mathcal{K}}$ its ℓ-adic completion. Then $\pi_*\hat{\mathcal{K}} \cong \mathbb{Z}_\ell[\beta, \beta^{-1}]$, where $\beta \in \pi_2\hat{\mathcal{K}}$ and multiplication by β defines an equivalence $S^2 \wedge \hat{\mathcal{K}} \simeq \hat{\mathcal{K}}$. The ring $\hat{\mathcal{K}}^0\hat{\mathcal{K}}$ of degree 0 stable operations in $\hat{\mathcal{K}}$ is naturally a profinite $\mathbb{Z}_\ell$-algebra; the profinite structure results from expressing this ring as $\lim_X \hat{\mathcal{K}}^0 X$ where X ranges through the finite subspectra of $\hat{\mathcal{K}}$. The closed subgroup $\Gamma' \subset (\hat{\mathcal{K}}^*\hat{\mathcal{K}})^\times$ of ring spectrum self-equivalences acts on $\pi_2\hat{\mathcal{K}} \cong \mathbb{Z}_\ell$, and the resulting homomorphism $c : \Gamma' \to \operatorname{Aut}(\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^\times$ is an isomorphism. Let Γ denote the kernel of the composite surjection $\Gamma' \to \mathbb{Z}_\ell^\times \to (\mathbb{Z}/\ell)^\times$ and $\Delta \cong (\mathbb{Z}/\ell)^\times$ the quotient group Γ'/Γ. Then Γ is abstractly isomorphic to the additive group of ℓ-adic integers, and there is a uniquely split extension

$$\{1\} \to \Gamma \to \Gamma' \to \Delta \to \{1\}\,.$$

Let Λ' denote $\mathbb{Z}_\ell[[\Gamma']]$ (this is isomorphic to $\lim_n \mathbb{Z}_\ell[(\mathbb{Z}/\ell^n)^\times]$). According to [26] or [33] there are isomorphisms

$$\hat{\mathcal{K}}^i(\hat{\mathcal{K}}) \cong \begin{cases} \Lambda' & i=0 \\ 0 & i=1 \end{cases}$$

We let $\mathbb{Z}_\ell(1)$ denote the Λ'-module $\pi_2\hat{\mathcal{K}}$ and $\mathbb{Z}_\ell(m)$ the m-fold tensor power of $\mathbb{Z}_\ell(1)$ (which for any $m \in \mathbb{Z}$ is isomorphic as a Λ'-module to $\pi_{2m}\hat{\mathcal{K}}$). If N is a Λ'-module, we denote by $N(m)$ the "m-fold Tate twist" $\mathbb{Z}_\ell(m) \otimes_{\mathbb{Z}_\ell} A$ of N. If F is a number field as in §2, we will use the canonical embedding $c^{-1} \cdot c_F$ to identify Γ'_F with a subgroup of Γ', Γ_F with a subgroup of Γ, and Λ'_F with a subring of Λ' and Λ_F with a subring of Λ. It is clear that under these identifications the notion of Tate twisting from §2 agrees with the one here.

4.4 Remark. For any spectrum X, multiplication with $\beta^{-1} \in \hat{\mathcal{K}}^2(S^0) = \pi_{-2}\hat{\mathcal{K}}$ induces isomorphisms $(\hat{\mathcal{K}}^iX)(-1) \cong \hat{\mathcal{K}}^{i+2}X$; the twisting comes from the formula $\gamma \cdot \beta = c(\gamma)^{-1}\beta$, $\gamma \in \Gamma'$. The group $\Gamma' \subset (\Lambda')^\times$ is isomorphic to the multiplicative group of classical ℓ-adic Adams operations under the correspondence $\gamma \mapsto \psi^{c(\gamma)}$.

Let $\mathcal{N}$ be the Moore spectrum $\mathcal{M}(\mathbb{Z}/\ell^\infty, -1) = \mathrm{colim}_k\, \Sigma^{-1}\mathcal{M}_{\ell^k}$. For any spectrum X, $X \wedge \mathcal{N}$ is the homotopy fibre of the map from X to its rationalization, and so $(X \wedge \mathcal{N})\hat{} \simeq \hat{X}$ and $\hat{\mathcal{K}}^*(X \wedge \mathcal{N}) \cong \hat{\mathcal{K}}^*(X)$. It is clear that $X \wedge \mathcal{N}$ is a *torsion spectrum* in the sense that its homotopy groups are torsion groups.

Recall that $(-)^\#$ denotes Pontriagin duality. The following lemma shows that $\hat{\mathcal{K}}^*$ is dual to an associated homology theory.

4.5 Lemma. *For any spectrum X there are natural isomorphisms*

$$\hat{\mathcal{K}}^iX \cong (\hat{\mathcal{K}}_{i-1}X \wedge \mathcal{N})^\# \ .$$

Proof. Let Y denote $X \wedge \mathcal{N}$ and C the cofibre of the spectrum completion map $\mathcal{K} \to \hat{\mathcal{K}}$. The homotopy groups of C vanish in odd dimensions and in each even dimension are isomorphic to the rational vector space $\mathbb{Z}_\ell/\mathbb{Z}_{(l)}$. Since the homotopy groups of Y are torsion, it follows that $C \wedge Y$ is contractible and that every map from a suspension of Y to C is null homotopic. Long exact sequences, then, show that the natural maps $\mathcal{K}_*Y \to \hat{\mathcal{K}}_*Y$ and $\mathcal{K}^*Y \to \hat{\mathcal{K}}^*Y$ are isomorphisms. As remarked above, the restriction map $\hat{\mathcal{K}}^*X \to \hat{\mathcal{K}}^*Y$ is also an isomorphism.

By the universal coefficient theorem for topological K-cohomology [1, 13.6], there are isomorphisms

$$\mathcal{K}^i(Y) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathcal{K}_{i-1}Y, \mathbb{Z}) \cong (\mathcal{K}_{i-1}Y)^\# \ .$$

Composing the various isomorphisms above gives the result. □

4.6 Proposition. *For any spectrum X there are isomorphisms*

$$\hat{\mathcal{K}}^iX \cong \lim_\alpha \hat{\mathcal{K}}^i(X_\alpha) \qquad i \in \mathbb{Z}\,.$$

where the limit is taken over all finite subspectra X_α of X.

Proof. Clearly $\hat{\mathcal{K}}_{i-1}(X \wedge \mathcal{N})$ is isomorphic to $\mathrm{colim}_\alpha\, \hat{\mathcal{K}}_{i-1}(X_\alpha \wedge \mathcal{N})$. The Pontriagin dual $\hat{\mathcal{K}}^iX$ of this group (4.5) is then given by the corresponding limit. □

4.7 Lemma. *Let f be a map of spectra. Then f is an equivalence with respect to $E_* = (\mathcal{K} \wedge \mathcal{M}_\ell)_*$ if and only if $\hat{\mathcal{K}}^*(f)$ is an isomorphism.*

Proof. Let C be the cofibre of f, and note that $\mathcal{K} \wedge \mathcal{M}_\ell$ is equivalent to $\hat{\mathcal{K}} \wedge \mathcal{M}_\ell$. If C is acyclic with respect to $(\hat{\mathcal{K}} \wedge \mathcal{M}_\ell)_*$, then by an inductive argument using the cofibration sequences

$$\mathcal{M}_\ell \to \mathcal{M}_{\ell^k} \to \mathcal{M}_{\ell^{k-1}}$$

the spectrum C is acyclic with respect to $(\hat{\mathcal{K}} \wedge \mathcal{M}_{\ell^k})_*$ $(k \geq 1)$. By passing to the limit, C is acyclic with respect to $\hat{\mathcal{K}} \wedge \mathcal{N})_*$. Therefore (4.5), $\hat{\mathcal{K}}^* X$ is zero.

Suppose that $\hat{\mathcal{K}}^*(X)$ is zero. By 4.5, $\hat{\mathcal{K}} \wedge \mathcal{N} \wedge X$ is contractible. The desired result follows from smashing the cofibration sequence

$$\Sigma^{-1}\mathcal{M}_\ell \to \mathcal{N} \xrightarrow{\ell} \mathcal{N}$$

with $\hat{\mathcal{K}}$ and X. $\square$

Let $L_{\mathcal{K}}$ denote Bousfield's localization functor with respect to $\mathcal{K}$ and $\hat{L}$ the localization functor with respect $\mathcal{K} \wedge \mathcal{M}_\ell$. In view of the above lemma we will refer to the functor $\hat{L}$ as "localization with respect to $\hat{\mathcal{K}}^*$". The following proposition describes $L_{\mathcal{K}}$ and gives the relationship between $L_{\mathcal{K}}$ and $\hat{L}$.

4.8 Proposition. [8, 4.7] [8, 2.11] *For any spectrum X there are canonical homotopy equivalences*

$$\begin{aligned} L_{\mathcal{K}}X &\simeq L_{\mathcal{K}}(S^0) \wedge X \\ \hat{L}X &\simeq L_{\mathcal{M}_\ell} L_{\mathcal{K}} X \simeq L_{\mathcal{M}_\ell}(L_{\mathcal{K}}(S^0) \wedge X) \end{aligned} .$$

There is a description of $L_{\mathcal{K}}(S^0)$ in [8, §4]. The localization $\hat{L}(X)$ has the surprising property that it is determined by the "germ of the spectrum X at infinity" in the following sense. Recall (1.13) that $P^k X$ denotes the k-connective cover of X.

4.9 Proposition. *For any spectrum X and integer k, the natural map $P^k X \to X$ induces an equivalence $\hat{L}(P^k X) \to \hat{L}(X)$.*

Proof. The fibre F of the map $P^k X \to X$ has homotopy groups which vanish above dimension k, and so F is equivalent to the colimit $\mathrm{colim}_{i<k} P_i F$ of its Postnikov stages. Each Postnikov stage $P_i F$ is built up from a finite number of fibrations involving Eilenberg-MacLane spectra, and so $\mathcal{K} \wedge \mathcal{M}_\ell \wedge P_i F$ is contractible [2]. It follows from a colimit argument that $\mathcal{K} \wedge \mathcal{M}_\ell \wedge F$ is contractible, and hence that $\hat{L}F \simeq *$. $\square$

The category of $\hat{\mathcal{K}}^*$-local spectra is by now extremely well understood, thanks to work of Bousfield [10] [11]. We will be dealing for the most part with exceptionally well behaved objects of this category.

4.10 Definition. A module N over a ring S (e.g. $S = \Lambda'$, Λ, Λ'_F, ...) is said to be *excellent* if it is finitely generated and has projective dimension at most 1. A spectrum X is said to be *excellent* if X is $\hat{\mathcal{K}}^*$-local and either $\hat{\mathcal{K}}^1 X = 0$ and $\hat{\mathcal{K}}^0 X$ is an excellent Λ'-module or vice versa.

4.11 Remark. The ring Λ' has global dimension 2, so the condition that a module N over Λ' be excellent is not extremely restrictive. Since $\Lambda' \cong \mathbb{Z}_\ell[\Delta] \otimes_{\mathbb{Z}_\ell} \Lambda$ and $\mathbb{Z}_\ell[\Delta]$ is isomorphic as a ring to a direct product of copies of $\mathbb{Z}_\ell$, the projective dimension of a module N over Λ' is the same as its projective dimension over Λ. Excellent modules over Λ are characterized in 7.7.

If X is the spectrum $(\hat{\mathcal{K}})^n$ $(n \geq 0)$ then for any spectrum Y the natural map

$$\hat{\mathcal{K}}^0(Y)^n \cong [Y, X] \longrightarrow \mathrm{Hom}_{\Lambda'}(\hat{\mathcal{K}}^0 X, \hat{\mathcal{K}}^0 Y) \cong \mathrm{Hom}_{\Lambda'}((\Lambda')^n, \hat{\mathcal{K}}^0 Y)$$

is an isomorphism. The same holds if X is a retract of $(\hat{\mathcal{K}})^n$. Suppose now that P is a finitely generated projective module over Λ'. The module P can be expressed as the image of an idempotent map $e : (\Lambda')^n \longrightarrow (\Lambda')^n$ for some n. The remark above shows that e can be realized by a map $\tilde{e} : (\hat{\mathcal{K}})^n \longrightarrow (\hat{\mathcal{K}})^n$ which is idempotent up to homotopy. By an elementary argument (cf. 9.4), the infinite mapping telescope X of $\tilde{e}$ is a spectrum which is a retract of $(\hat{\mathcal{K}})^n$ and has the property that $\hat{\mathcal{K}}^0 X \cong P$. The spectrum X is $\hat{\mathcal{K}}^*$-local (because the class of $\hat{\mathcal{K}}^*$-local spectra is closed under retracts) and consequently excellent. This reasoning leads to the following lemma.

4.12 Lemma. *If P is a finitely generated projective module over Λ', then there exists up to homotopy a unique excellent spectrum $X = \mathcal{M}_{\mathcal{K}}(P, 0)$ such that $\hat{\mathcal{K}}^0 X = P$. Moreover, for any spectrum Y the natural map*

$$[Y, X] \to \mathrm{Hom}_{\Lambda'}(\hat{\mathcal{K}}^0 X, \hat{\mathcal{K}}^0 Y)$$

is an isomorphism.

Suppose now that X is an excellent spectrum, say with $\hat{\mathcal{K}}^1 X = 0$ and $\hat{\mathcal{K}}^0 X = N$. By 4.12, any finitely generated projective resolution

$$N \leftarrow P_0 \leftarrow P_1 \tag{4.13}$$

of N can be realized by a cofibre sequence

$$X \longrightarrow \mathcal{M}_{\mathcal{K}}(P_0, 0) \longrightarrow \mathcal{M}_{\mathcal{K}}(P_1, 0) \tag{4.14}$$

(in the sense that the original resolution can be recovered by applying $\hat{\mathcal{K}}^0$ to the cofibre sequence). A short calculation with 4.14, 4.13 and 4.12 leads to the following proposition.

4.15 Proposition. *Let X be an excellent spectrum with, say, $\hat{\mathcal{K}}^1 X = 0$. Then for any spectrum Y with $\hat{\mathcal{K}}^1 Y = 0$ there are natural isomorphisms*

$$[\Sigma^i Y, X] \cong \begin{cases} \mathrm{Hom}_{\Lambda'}(\hat{\mathcal{K}}^0 X, (\hat{\mathcal{K}}^0 Y)(m)) & i = 2m \\ \mathrm{Ext}^1_{\Lambda'}(\hat{\mathcal{K}}^0 X, (\hat{\mathcal{K}}^0 Y)(m)) & i = 2m - 1 \end{cases}$$

4.16 Remark. Taking $Y = X$ in 4.15 gives $[X, X] = \mathrm{End}_{\Lambda'}(\hat{\mathcal{K}}^0 X)$. Since Λ' is commutative, it follows that an excellent spectrum X has an "internal" action of Λ', in the sense that there is a homomorphism $\phi : \Lambda' \to [X, X]$ such that $\phi(\lambda)^* : \hat{\mathcal{K}}^0 X \to \hat{\mathcal{K}}^0 X$ is multiplication by λ. Taking Y to be an appropriate sphere in 4.10 gives the formulas

$$\pi_i X \cong \begin{cases} \mathrm{Hom}_{\Lambda'}(\hat{\mathcal{K}}^0 X, \mathbb{Z}_\ell(m)) & i = 2m \\ \mathrm{Ext}^1_{\Lambda'}(\hat{\mathcal{K}}^0 X, \mathbb{Z}_\ell(m)) & i = 2m - 1 \end{cases}$$

Given an excellent Λ'-module N, it is clearly possible to realize a free resolution (4.13) of N by a cofibration sequence (4.14). Taking suspensions and using 4.15 for uniqueness gives the following result.

4.17 Proposition. *If N is an excellent Λ' module, then for any integer i there exists up to homotopy a unique excellent spectrum $X = \mathcal{M}_{\mathcal{K}}(N, i)$ such that $\hat{\mathcal{K}}^i(X)$ is isomorphic to N as a module over Λ'.*

Remark. The spectrum $\mathcal{M}_{\mathcal{K}}(N, i)$ is a "Moore spectrum" with respect to $\hat{\mathcal{K}}^*$.

Let $\mathbb{F}$ be the finite field from §2, so that $\mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F}) \cong \Gamma'_F$.

4.18 Proposition. *Let N denote the Λ'-module $\Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_\ell(0) = \mathbb{Z}_\ell[\Gamma'/\Gamma'_F]$. Then $\hat{L}(K\mathbb{F})$ is the excellent spectrum $\mathcal{M}_{\mathcal{K}}(N, 0)$.*

Proof. Let $|\mathbb{F}| = q$ and fix an embedding $\bar{\mathbb{F}}^\times \subset \mathbb{C}^\times$ which, after restricting to μ_{ℓ^∞} corresponds under the reduction map $R_\infty \to \mathbb{F}_\infty$ to the one obtained by the inclusion $R_\infty \subset \mathbb{C}$. Quillen's Brauer lift [34] defines an ℓ-equivalence $\Theta : \mathrm{BGL}(\mathbb{F})^+ \to F\Psi^q$, where $F\Psi^q$ is the fibre of $(\psi^q - \mathrm{id}) : \mathrm{BU} \to \mathrm{BU}$. After ℓ-adic completion Θ is an infinite loop map [28] and there results a cofibre sequence

$$P^0\hat{K}\mathbb{F} \to P^0\hat{\mathcal{K}} \xrightarrow{\psi^q - \mathrm{id}} P^0\hat{\mathcal{K}}$$

which by 4.9 is the 0-connective cover of its $\hat{\mathcal{K}}^*$-localization

$$\hat{L}(K\mathbb{F}) \to \hat{\mathcal{K}} \xrightarrow{\psi^q - \mathrm{id}} \hat{\mathcal{K}}\,. \tag{4.19}$$

Since by choice of $\mathbb{F}$ the operation ψ^q is a topological generator of $\Gamma'_F \subset \Gamma'$ (4.4), an inspection of the $\hat{\mathcal{K}}^*$-cohomology sequence associated to 4.19 proves the proposition. □

4.20 Remark. A calculation with 4.15 shows that the maps $\hat{L}(K\mathbb{F}_n) \to \hat{\mathcal{K}}$ constructed as above are compatible for various n and are Γ'-equivariant, where Γ' acts on $\hat{L}(K\mathbb{F}_n)$ via the internal action of Λ' (4.16). It is easy to see from the definition of Brauer lift that the internal action of $\Gamma'_F \subset \Gamma'$ on $\hat{L}(K\mathbb{F}_n)$ agrees with the Galois action induced by the operation of Γ'_F on $\mathbb{F}_n$ by ring automorphisms. The maps $\hat{L}(K\mathbb{F}_n) \to \hat{\mathcal{K}}$ are maps of ring spectra, and passing to the limit in n gives a map of ring spectra $\hat{L}(\mathrm{colim}_n\, \hat{L}(K\mathbb{F}_n)) \cong \hat{L}(K\mathbb{F}_\infty) \to \hat{\mathcal{K}}$ which is an equivalence.

4.21 Module spectra over $\hat{L}(K\mathbb{F})$. The map

$$\mathbb{Z}_\ell[[\Gamma']] = \Lambda' = \hat{\mathcal{K}}^0\hat{\mathcal{K}} \longrightarrow \hat{\mathcal{K}}^0(\hat{\mathcal{K}} \wedge \hat{\mathcal{K}}) = \Lambda'\hat{\otimes}\Lambda' = \mathbb{Z}_\ell[[\Gamma' \times \Gamma']]$$

given by the ring spectrum multiplication $\hat{\mathcal{K}} \wedge \hat{\mathcal{K}} \to \hat{\mathcal{K}}$ is induced by the diagonal map $\Gamma' \to \Gamma' \times \Gamma'$. It follows from the above and naturality that the comultiplication

$$\mathbb{Z}_\ell[\Gamma'/\Gamma'_F] = \hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F})) \longrightarrow \hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F}) \wedge \hat{L}(K\mathbb{F})) = \mathbb{Z}_\ell[(\Gamma'/\Gamma'_F) \times (\Gamma'/\Gamma'_F])$$

given by the ring structure of $\hat{L}(K\mathbb{F})$ is induced by the diagonal map $\Gamma'/\Gamma'_F \to (\Gamma'/\Gamma'_F)\times(\Gamma'/\Gamma'_F)$. This comultiplication gives a coalgebra structure to $\hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F}))$. The counit map $\epsilon : \hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F})) \longrightarrow \mathbb{Z}_\ell$ can be identified with the augmentation $\mathbb{Z}_\ell[\Gamma'/\Gamma'_F] \to \mathbb{Z}_\ell$.

4.22 Proposition. *Suppose that X is an excellent spectrum, with, say, $\hat{\mathcal{K}}^1X = 0$. Then the following three sets are in natural bijective correspondence:*

(1) *$\hat{L}(K\mathbb{F})$-module structures on X,*
(2) *Λ'-module maps $f : \hat{\mathcal{K}}^0X \longrightarrow \hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F}))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}^0(X)$ which satisfy the obvious coassociativity identity as well as the corresponding counit identity, and*
(3) *isomorphism classes of pairs (N, j), where N is a module over Λ'_F and $j : \Lambda' \otimes_{\Lambda'_F} N \longrightarrow \hat{\mathcal{K}}^0X$ is an isomorphism.*

Proof of 4.22. A bijection between (1) and (2) arises by 4.15 from taking induced maps on $\hat{\mathcal{K}}^0$. Giving a pair (N, j) as in (3), then there is an evident Λ'-module map

$$\hat{\mathcal{K}}^0X \cong \Lambda' \otimes_{\Lambda'_F} N \longrightarrow \mathbb{Z}_\ell[\Gamma'/\Gamma'_F] \otimes_{\mathbb{Z}_\ell} (\Lambda' \otimes_{\Lambda'_F} N) = \hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F})) \otimes_{\mathbb{Z}_\ell} \hat{\mathcal{K}}^0X$$

induced by the diagonal $\Gamma' \longrightarrow (\Gamma'/\Gamma'_F) \times \Gamma'$. This map satisfies the coassociativity and counit identities necessary for (2). Suppose on the other hand that there is given a map f as in (2). Let $N = \{x \in \hat{\mathcal{K}}^0X \mid f(x) = e\otimes x\}$, where $e \in \Gamma'/\Gamma'_F$ is the identity coset. Then N is a Γ'_F-module and it is straightforward to check using the coassociativity of f that the natural map $\Lambda' \otimes_{\Lambda'_F} N \longrightarrow \hat{\mathcal{K}}^0X$ is an isomorphism. □

§5. A module spectrum structure.

In this section we will prove the existence of a $\hat{\mathcal{K}}$-module structure on a spectrum closely related to $\hat{L}(K^{\mathrm{red}}R_\infty)$. This is crucial for the arguments in §6. We also obtain wedge decompositions (5.5) for the spectra $\hat{L}(KR_n)$

Two maps $f, g : X \longrightarrow Y$ of spectra are said to agree *up to weak homotopy* (or sometimes up to visible homotopy) if they agree up to homotopy on any finite subspectrum of X, or equivalently if the difference $(f - g)$ is a phantom map. A square *weakly commutes* if the two composite maps involved agree up to weak homotopy.

Recall the Moore spectrum $\mathcal{N} = \mathcal{M}(\mathbb{Z}/\ell^\infty, -1) = \operatorname{colim}_k \Sigma^{-1}\mathcal{M}_{\ell^k}$ introduced in §4.

5.1 Theorem. *For each* $0 \le n \le \infty$ *the spectrum* $\hat{L}(K^{\mathrm{red}}R_n) \wedge \mathcal{N}$ *can be given the structure of a module spectrum over* $\hat{L}(K\mathbb{F}_n)$*. Furthermore, these structures can be chosen so that for each* n*, each* $m \ge n$ *and each* $\gamma \in \Gamma'_F$ *the diagrams*

$$\begin{array}{ccc} \hat{L}(K\mathbb{F}_n) \wedge \hat{L}(K^{\mathrm{red}}R_n) \wedge \mathcal{N} & \longrightarrow & \hat{L}(K\mathbb{F}_m) \wedge \hat{L}(K^{\mathrm{red}}R_m) \wedge \mathcal{N} \\ \downarrow & & \downarrow \\ \hat{L}(K^{\mathrm{red}}R_n) \wedge \mathcal{N} & \longrightarrow & \hat{L}(K^{\mathrm{red}}R_m) \wedge \mathcal{N} \end{array}$$

$$\begin{array}{ccc} \hat{L}(K\mathbb{F}_m) \wedge \hat{L}(K^{\mathrm{red}}R_m) \wedge \mathcal{N} & \xrightarrow{\gamma\wedge\gamma} & \hat{L}(K\mathbb{F}_m) \wedge \hat{L}(K^{\mathrm{red}}R_m) \wedge \mathcal{N} \\ \downarrow & & \downarrow \\ \hat{L}(K^{\mathrm{red}}R_m) \wedge \mathcal{N} & \xrightarrow{\gamma} & \hat{L}(K^{\mathrm{red}}R_m) \wedge \mathcal{N} \end{array}$$

commute for $m < \infty$ *and weakly commute for* $m = \infty$*.*

5.2 Remark. Later on (9.7) we will extend 5.1 to the case $n = -1$. By 4.20, Theorem 5.1 implies that $\hat{L}(K^{\mathrm{red}}R_\infty) \wedge \mathcal{N}$ is a module spectrum over $\hat{\mathcal{K}}$.

5.3 Remark. In light of 4.3 (with L_E taken to be ℓ-completion), Theorem 5.1 provides each spectrum $\hat{L}(K^{\mathrm{red}}R_n)$ $(n \ge 0)$ with a structure of module spectrum over $\hat{L}(K\mathbb{F}_n)$. The diagrams in 5.1 commute after $\hat{\mathcal{K}}^*$ is applied, even if $m = \infty$ or the smash factors of $\mathcal{N}$ are removed (4.6).

The proof of 5.1 depends on a localized stable version of the basic splitting theorem from [18]. Suppose that A is a commutative ring and let $\mu(A)$ denote the group of ℓ-primary roots of unity in A. Recall that there is a natural map of ring spectra

$$f(A) : \mathrm{B}\mu(A)_+ \to KA$$

such that $\Omega_0^\infty f(A)$ is induced by the usual map $\mathrm{B}\mu(A) \to \mathrm{B}A^\times \to \mathrm{BGL}(A)$. Let $r_n = f(R_n)$, $f_n = f(\mathbb{F}_n)$. Note that $\mu(R_n) \cong \mu(\mathbb{F}_n)$. Let $\pi_n : KR_n \to K\mathbb{F}_n$ be the map induced by the quotient map $R_n \to \mathbb{F}_n$.

5.4 Theorem. *For* $n \ge 0$ *there is a unique map* $h_n : \hat{L}(K\mathbb{F}_n) \to \hat{L}(KR_n)$ *such that the composite* $h_n \cdot \hat{L}f_n$ *is* $\hat{L}r_n$*. Moreover*

(1) *The composite* $\hat{L}\pi_n \cdot h_n$ *is the identity map of* $\hat{L}(K\mathbb{F}_n)$*.*
(2) h_n *is a map of ring spectra.*
(3) h_n *is equivariant with respect to the actions of the group* Γ'_F *on* $\hat{L}(K\mathbb{F}_n)$ *and* $\hat{L}(KR_n)$*.*
(4) *The following diagrams, in which the vertical maps are induced by the obvious ring inclusions, commute:*

$$\begin{array}{ccc} \hat{L}(K\mathbb{F}_n) & \xrightarrow{h_n} & \hat{L}(KR_n) \\ \downarrow & & \downarrow \\ \hat{L}(K\mathbb{F}_{n+1}) & \xrightarrow{h_{n+1}} & \hat{L}(KR_{n+1}) \end{array} .$$

5.5 Remark. Theorem 5.4 implies that there is a wedge decomposition of $\hat{L}(KR_n)$ as $\hat{L}(K\mathbb{F}_n) \vee \hat{L}(K^{\mathrm{red}}R_n)$.

5.6 Remark. By Suslin [42] there is a natural homotopy equivalence $\hat{L}(K\mathbb{C}) \simeq \hat{\mathcal{K}}$. Let $i_n : \hat{L}(KR_n) \to \hat{L}(K\mathbb{C}) \simeq \hat{\mathcal{K}}$ be the map induced by the inclusion $R_n \subset \mathbb{C}$. A character calculation which depends on the definition of Brauer lift shows that the composite $i_n \cdot h_n$ is the usual Brauer lifting map $\hat{L}(K\mathbb{F}_n) \to \hat{\mathcal{K}}$ (4.19).

One of the main ingredients in the proof of 5.4 is the following remarkable result of Bousfield.

5.7 Proposition. [11, 2.3] *There is a functor Φ_ℓ from the homotopy category of spaces to the homotopy category of spectra such that, for any spectrum X, there is a natural homotopy equivalence $\Phi_\ell(\Omega_0^\infty X) \to \hat{L}X$.*

5.8 Proposition. *For $n \geq 0$ there exist maps $s_n : \hat{L}(K\mathbb{F}_n) \to \hat{L}(\mathrm{B}\mu(\mathbb{F}_n))$ such that in the diagram*

$$\hat{L}(K\mathbb{F}_n) \xrightarrow{s_n} \hat{L}(\mathrm{B}\mu(\mathbb{F}_n)) \xrightarrow{\hat{L}f_n} \hat{L}(K\mathbb{F}_n) \xrightarrow{s_n} \hat{L}(\mathrm{B}\mu(\mathbb{F}_n)) \xrightarrow{\hat{L}r_n} \hat{L}(KR_n)$$

the following two conditions hold.

(1) *The composite $(\hat{L}f_n) \cdot s_n$ is the identity map of $\hat{L}(K\mathbb{F}_n)$, and*
(2) *the composite $(\hat{L}r_n) \cdot s_n \cdot (\hat{L}f_n)$ is homotopic to the map $\hat{L}r_n$.*

Proof. Fix n, let $\mu = \mu(\mathbb{F}_n)$ and let S denote the ring $\mathbb{Z}(\mu)$ of cyclotomic integers. Let E denote the residue field of S which is the image of the composite $S \to R_n \to \mathbb{F}_n$, so that $\mu(S) = \mu(E) = \mu(R_n) = \mu(\mathbb{F}_n)$. By [18, 4.1] there is a map $s : \Omega_0^\infty KE \to \Omega_0^\infty(\mathrm{B}\mu_+)$ such that in the diagram

$$\Omega_0^\infty KE \xrightarrow{s} \Omega_0^\infty(\mathrm{B}\mu_+) \xrightarrow{\Omega_0^\infty f(E)} \Omega_0^\infty KE \xrightarrow{s} \Omega_0^\infty(\mathrm{B}\mu_+) \xrightarrow{\Omega_0^\infty f(S)} KS$$

the following two conditions hold.

(1) The composite $\Omega_0^\infty f(E) \cdot s$ is homotopic to the identity map of $\Omega_0^\infty KE$, and
(2) the composite $\Omega_0^\infty f(S) \cdot s \cdot \Omega_0^\infty f(E)$ is homotopic to $\Omega_0^\infty f(S)$.

The inclusion $E \to \mathbb{F}_n$ induces an equivalence $KE \to K\mathbb{F}_n$ (use [34], note that the spectra by convention are localized at ℓ, and recall that the prime $\mathcal{P}$ was chosen in §2 so that $q = |\mathbf{F}_{-1}|$ is a topological generator of $c_F(\Gamma'_F) \subset \mathbb{Z}_\ell^\times$). The desired map s_n can be obtained by applying Bousfield's functor Φ_ℓ to the map s. □

Proof of 5.4. Let s_n be the map of 5.8 and choose $h_n = \hat{L}r_n \cdot s_n$. The identity $h_n \cdot \hat{L}f_n = \hat{L}r_n$ is then 5.8(2). By 5.8(1) $\hat{L}f_n$ is a retraction (i.e. has a right inverse s_n), so h_n is actually determined uniquely by this identity. The remaining statements follow routinely (compare [18, 4.9]) from the fact that $\hat{L}f_n$ is a retraction. To prove (1), for instance compute

$$\hat{L}\pi_n \cdot h_n \cdot \hat{L}f_n = \hat{L}\pi_n \cdot \hat{L}r_n = \hat{L}f_n$$

and compose on the right with s_n. For (2) observe in addition that $\hat{L}f_n$ and $\hat{L}r_n$ are maps of ring spectra, and for (3) that $\hat{L}f_n$ and $\hat{L}r_n$ are equivariant with respect to the actions of Γ'_F on the spectra involved. □

Let S be a ring spectrum with multiplication $m : S\wedge S \to S$ and unit $i : S^0 \to S$. Let X be a spectrum. A *weak S-module structure* on X is a multiplication map $m_X : S \wedge X \to X$ such that the diagrams

$$\begin{array}{ccc} S\wedge S\wedge X & \xrightarrow{\mathrm{id}\wedge m_X} & S\wedge X \\ {\scriptstyle m\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle m_X} \\ S\wedge X & \xrightarrow{m_X} & X \end{array} \qquad \begin{array}{ccc} S^0\wedge X & \xrightarrow{i\wedge\mathrm{id}} & S\wedge X \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle m_X} \\ X & \xrightarrow{\mathrm{id}} & X \end{array}$$

commute up to weak homotopy. A *rigidification* of such a weak module structure is a genuine S-module structure $m'_X : S \wedge X \to X$ such that m'_X agrees with m_X up to weak homotopy.

5.9 Lemma. *Any weak $\hat{\mathcal{K}}$-module spectrum structure on a torsion spectrum X can be rigidified.*

Proof. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be Moore spectra of type $(\pi_0 X, 0)$ and $(\pi_1 X, 1)$ respectively, and let $\mathcal{M} = \mathcal{M}_0 \vee \mathcal{M}_1$. The Hurewicz map $\pi_i\mathcal{M} \to \pi_i(\mathcal{K}\wedge\mathcal{M})$ is an isomorphism in dimensions 0 and 1, and there exists a map $j : \mathcal{M} \to X$ inducing an isomorphism on homotopy in dimensions 0 and 1. A weak $\hat{\mathcal{K}}$-module structure on X allows j to be extended to a map $j' : \hat{\mathcal{K}} \wedge \mathcal{M} \to X$, which is again an isomorphism on π_0 and π_1 and hence an equivalence, since the homotopy groups involved are 2-fold periodic. The smash product $\hat{\mathcal{K}}\wedge\mathcal{M}$ has an obvious $\hat{\mathcal{K}}$-module structure, which can be transported to X by the equivalence j' and gives the desired rigidification. □

5.10 Lemma. *The functors $L_{\mathcal{K}}(-)$ and $\hat{L}(-)\wedge\mathcal{N}$ commute with filtered colimits of spectra.*

Proof. It follows from 4.8 that there are homotopy equivalences

$$\hat{L}(X)\wedge\mathcal{M}_\ell \simeq L_{\mathcal{K}}(X)\wedge\mathcal{M}_\ell \simeq X\wedge L_{\mathcal{K}}(S^0)\wedge\mathcal{M}_\ell\,.$$

An inductive argument gives similar equivalences with $\mathcal{M}_\ell$ replaced by $\mathcal{M}_{\ell^k}$, and passing to the colimit gives an equivalence $\hat{L}(X)\wedge\mathcal{N} \simeq X\wedge L_{\mathcal{K}}(S^0)\wedge\mathcal{N}$. The lemma follows from the fact that smash product commutes with filtered colimits. □

5.11 Lemma. *Let $X_1 \to X_2 \to \cdots$ be a direct system of spectra indexed by the positive integers (the maps here are genuine maps of spectra, not just maps taken up to homotopy). Then for any spectrum Y the natural map $[\operatorname{colim}_i X_i, Y] \to \lim_i[X_i, Y]$ is a surjection.*

Proof. This is the spectral version of the Milnor sequence [13, IX, 3.3]; the kernel of the surjection is $\lim^1_i[\Sigma X_i, Y]$. Let W be the coproduct (wedge) of the spectra X_i and $s : W \to W$ the map which shifts each wedge factor into the next with the bonding map from the direct system. It is possible (by calculation of homotopy groups) to identify $\operatorname{colim}_i X_i$ as the cofibre of $s - \mathrm{id} : W \to W$, and then to derive the Milnor sequence from the long exact homotopy sequence obtained by mapping this cofibration sequence to Y. □

Proof of 5.1. It follows from 5.4(2) that for any $0 \le n < \infty$ the spectrum $\hat{L}(KR_n)\wedge\mathcal{N}$ is a module spectrum over $\hat{L}(K\mathbb{F}_n)$. From 5.4(1) it follows that $\hat{L}(K^{\mathrm{red}}R_n)\wedge\mathcal{N}$

is also a module spectrum over $\hat{L}(K\mathbb{F}_n)$. Let Y denote $\operatorname{colim}_n \hat{L}(K^{\mathrm{red}}R_n)\wedge\mathcal{N}$ and let X denote $\operatorname{colim}_n \hat{L}(K\mathbb{F}_n)$. Then $Y \simeq \hat{L}(K^{\mathrm{red}}R_\infty)\wedge\mathcal{N}$ (5.10), and passing to the limit over n with 5.4(4) and 5.11 shows that Y is a weak module spectrum over X. As in 4.3, this implies that $Y = L_{\mathcal{K}}Y$ (5.10) is a weak module spectrum over $L_{\mathcal{K}}X$. The map $L_{\mathcal{K}}X \to \hat{L}X = (L_{\mathcal{K}}X)\hat{}$ is a mod ℓ equivalence and so has a rational cofibre; it follows that for any torsion spectrum such as Y the natural map $L_{\mathcal{K}}X\wedge Y \to \hat{L}X\wedge Y$ is an equivalence. This implies that Y is a weak module spectrum over $\hat{L}X \simeq \hat{\mathcal{K}}$ (4.20) and hence by 5.9 a module spectrum over $\hat{\mathcal{K}}$. The commutativity of the diagrams in 5.1 for $0\le n\le m<\infty$ follows from 5.4(3), and the weak commutativity for $m=\infty$ by passing to the limit. □

§6. Proof of the main theorem.

In this section we give a proof of 1.7, and point out some functoriality properties (6.12 ff.) of the isomorphisms it gives. The starting point is the following deep theorem of Thomason. Recall that $\mathcal{M}_{\ell^k}$ denotes a Moore spectrum of type $(\mathbb{Z}/\ell^k,0)$. We adopt the convention that $\mathbb{Z}/\ell^k(j/2)$ is the trivial Γ'-module if j is odd (§3).

6.1 Theorem. [43, 4.1, A.14] *Suppose that S is a regular ring containing $1/\ell$ and satisfying certain mild étale cohomological conditions; for instance, S could be the ring of ℓ-integers in a number field, a union (e.g. R_∞) of such rings, a finite field of characteristic different from ℓ, or a union (e.g. $\mathbb{F}_\infty$) of such finite fields. Then there are natural left half plane spectral sequences of homological type*

$$E^2_{i,j} = \mathrm{H}^{-i}_{\text{ét}}(S;\mathbb{Z}/\ell^k(j/2)) \Rightarrow \pi_{i+j}\hat{L}(KS)\wedge\mathcal{M}_{\ell^k}\,.$$

In the cases we are interested in, Thomason's spectral sequence collapses.

6.2 Proposition. *If S is one of the rings R_∞, $\mathbb{F}$ or $\mathbb{F}_\infty$, then there are natural isomorphisms*

$$\pi_{2m-2}\hat{L}(KS)\wedge\mathcal{N} \cong \mathrm{H}^1_{\text{ét}}(S;\mathbb{Z}/\ell^\infty(m))$$
$$\pi_{2m-3}\hat{L}(KS)\wedge\mathcal{N} \cong \mathrm{H}^0_{\text{ét}}(S;\mathbb{Z}/\ell^\infty(m-1))$$

for all m. If S is the ring R there are similar natural isomorphisms for all $m\ge 2$.

6.3 Remark. By 3.6, if S is $\mathbb{F}$ or $\mathbb{F}_\infty$ the groups above involving $\mathrm{H}^1_{\text{ét}}(S;\mathbb{Z}/\ell^\infty(m))$ vanish for $m\neq 0$.

Proof of 6.2. By 3.1, the groups $\mathrm{H}^i_{\text{ét}}(S;\mathbb{Z}/\ell^k(m))$ vanish for $i>2$. Thomason's spectral sequence therefore gives isomorphisms

$$\pi_{2m-1}\hat{L}(KS)\wedge\mathcal{M}_{\ell^k} \cong \mathrm{H}^1_{\text{ét}}(S;\mathbb{Z}/\ell^k(m))$$

and short exact sequences

$$0\to \mathrm{H}^2_{\text{ét}}(S;\mathbb{Z}/\ell^k(m)) \to \pi_{2m-2}\hat{L}(KS)\wedge\mathcal{M}_{\ell^k} \to \mathrm{H}^0_{\text{ét}}(S;\mathbb{Z}/\ell^k(m-1))\to 0\,.$$

Now by 3.6 and 3.7, for the allowed values of m (i.e. $m\ge 2$ if $S=R$) the groups $\mathrm{H}^2_{\text{ét}}(S;\mathbb{Z}/\ell^\infty(m))$ vanish. The desired formula thus results from taking a colimit over k of the above and introducing a shift to compensate for the fact that $\mathcal{N}$ is the desuspension of the colimit of the $\mathcal{M}_{\ell^k}$. $\square$

6.4 Remark. Taking a limit over k with 6.1 and using 3.4 shows that if S is the ring of ℓ-integers in a number field then there are natural isomorphisms

$$\pi_{2m-1}\hat{L}(KS) \cong \mathrm{H}^1_{\text{ét}}(S; \mathbb{Z}_\ell(m))$$
$$\pi_{2m-2}\hat{L}(KS) \cong \mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(m))$$

for all m in the case of the first formula and for $m \neq 1$ in the case of the second. For $i \geq 1$ these are the formulas for $\pi_i \hat{K}S$ predicted by the Lichtenbaum-Quillen conjecture, and in fact (1.11) the conjecture is equivalent to the conjecture that the natural map

$$\pi_i \hat{K}S \to \pi_i \hat{L}(\hat{K}S) \cong \pi_i \hat{L}(KS) \tag{6.5}$$

is an isomorphism for $i \geq 1$. This map is known to be an isomorphism for $i = 1$ and $i = 2$ [15, 8.2]. However by 6.1 there is a short exact sequence

$$0 \to \mathrm{H}^2_{\text{ét}}(S; \mathbb{Z}_\ell(1)) \to \pi_0 \hat{L}(KS) \to \mathrm{H}^0_{\text{ét}}(S; \mathbb{Z}_\ell(0)) \to 0$$

(where $\mathrm{H}^0_{\text{ét}}(S; \mathbb{Z}_\ell(0)) \cong \mathbb{Z}_\ell$)) so that, in view of 3.10, if S has more than one prime above ℓ the map in 6.5 is definitely not an isomorphism for $i = 0$. A naturality argument with 3.10 shows that in the case $S = R_n$ there are isomorphisms

$$\pi_i \hat{L}(K^{\text{red}} R_n) \cong \begin{cases} E'_n(\text{red}) & i = 1 \\ \mathrm{H}^2_{\text{ét}}(R_n; \mathbb{Z}_\ell(1)) & i = 0 \end{cases}.$$

Recall that γ'_F is a chosen topological generator of Γ'_F.

6.6 Proposition. *For any integer m there are Γ'_F-module isomorphisms*

$$\begin{aligned} \pi_{2m-2}\hat{L}(K^{\text{red}} R_\infty) \wedge \mathcal{N} &\cong \mathrm{H}^1_{\text{ét}}(R_\infty; \mathbb{Z}/\ell^\infty(m)) \\ \pi_{2m-3}\hat{L}(K^{\text{red}} R_\infty) \wedge \mathcal{N} &\cong 0 \end{aligned}.$$

Moreover there is a natural fibration sequence

$$\hat{L}(K^{\text{red}} R) \wedge \mathcal{N} \to \hat{L}(K^{\text{red}} R_\infty) \wedge \mathcal{N} \xrightarrow{\text{id} - \gamma'_F} \hat{L}(K^{\text{red}} R_\infty) \wedge \mathcal{N}.$$

6.7 Remark. Taking the ℓ-completion of the fibration sequence in 6.6 gives a fibration sequence

$$\hat{L}(K^{\text{red}} R) \to \hat{L}(K^{\text{red}} R_\infty) \xrightarrow{\text{id} - \gamma'_F} \hat{L}(K^{\text{red}} R_\infty).$$

This is a reflection in a special case of Thomason's theorem that $\hat{\mathcal{K}}^*$-localized algebraic K-theory satisfies étale descent.

Proof of 6.6. There is a natural strictly commutative diagram of spectra (i.e., a diagram not just commutative up to homotopy) which is homotopy equivalent to the following diagram

$$\begin{array}{ccccc} \hat{L}(KR)\wedge\mathcal{N} & \longrightarrow & \hat{L}(KR_\infty)\wedge\mathcal{N} & \xrightarrow{\mathrm{id}-\gamma'_F} & \hat{L}(KR_\infty)\wedge\mathcal{N} \\ \downarrow & & \downarrow & & \downarrow \\ \hat{L}(K\mathbb{F})\wedge\mathcal{N} & \longrightarrow & \hat{L}(K\mathbb{F}_\infty)\wedge\mathcal{N} & \xrightarrow{\mathrm{id}-\gamma'_F} & \hat{L}(K\mathbb{F}_\infty)\wedge\mathcal{N} \end{array}$$

and in which the horizontal composites are strictly trivial. Since the natural restriction maps

$$\mathrm{H}^0_{\text{ét}}(R_\infty;\mathbb{Z}/\ell^\infty(m)) \to \mathrm{H}^0_{\text{ét}}(\mathbb{F}_\infty;\mathbb{Z}/\ell^\infty(m))$$
$$\mathrm{H}^0_{\text{ét}}(R;\mathbb{Z}/\ell^\infty(m)) \to \mathrm{H}^0_{\text{ét}}(\mathbb{F};\mathbb{Z}/\ell^\infty(m))$$

are isomorphisms for all m, it follows from 6.2 and 6.3 that the vertical homotopy group maps induced by the above diagram are isomorphisms in dimension i for i odd and have zero target in dimension i for i even, where for the column on the far left we have to add the condition $i > 1$. Taking vertical fibres then gives a fibration diagram (cf. [4, Lemma 1.2])

$$\hat{L}(K^{\mathrm{red}}R)\wedge\mathcal{N} \to \hat{L}(K^{\mathrm{red}}R_\infty)\wedge\mathcal{N} \xrightarrow{\mathrm{id}-\gamma'_F} \hat{L}(K^{\mathrm{red}}R_\infty)\wedge\mathcal{N} \qquad (6.8)$$

in which the homotopy groups of the spectra involved are easy to compute by long exact sequences. □

A module A over a profinite group G is said to be *discrete* if every element of A is fixed by a closed subgroup in G of finite index. Let $\mathcal{U}$ denote the right adjoint of the forgetful functor from discrete ℓ-torsion Γ'-modules to ℓ-torsion abelian groups. Explicitly, $\mathcal{U}(A)$ is the group of continuous A-valued functions on Γ', with Γ' acting by $(\gamma f)(x) = f(x\gamma)$. Let $\mathcal{U}_F$ denote the right adjoint of the forgetful functor from discrete ℓ-torsion Γ'-modules to discrete ℓ-torsion Γ'_F-modules. Explicitly, $\mathcal{U}_F(A)$ is the group of continuous Γ'_F-equivariant A-valued functions on Γ'. If A is a discrete ℓ-torsion Γ'_F module, then there is a natural action of Γ'_F on $\mathcal{U}(A)$ given by the formula $(\gamma\cdot f)(x) = \gamma(f(\gamma^{-1}x))$, where $x\in\Gamma'$ and $f:\Gamma'\to A$ is a continuous function.

6.9 Lemma. *For any discrete ℓ-torsion Γ'_F-module A there is a natural short exact sequence of Γ'-modules:*

$$0\to\mathcal{U}_F(A)\to\mathcal{U}(A)\xrightarrow{\mathrm{id}-\gamma'_F}\mathcal{U}(A)\to 0\,.$$

Proof. For a function $f\in\mathcal{U}(A)$ to be in the kernel of $\mathrm{id}-\gamma'_F$, it must be true that, for all $x\in\Gamma'$, $f(x) = \gamma'_F(f((\gamma'_F)^{-1}x))$ or $(\gamma'_F)^{-1}f(x) = f((\gamma'_F)^{-1}x)$. Since f is continuous and γ'_F is a topological generator of Γ'_F, these equalities hold if and only if f is Γ'_F-equivariant, i.e., $f\in\mathcal{U}_F(A)$.

It remains to show that $(\mathrm{id} - \gamma'_F)$ is surjective. Let $\mathcal{V}(A)$ denote the abelian group $\mathcal{U}(A)$ furnished with the action of Γ'_F given by $(\gamma \cdot f)(x) = f(\gamma^{-1}x)$, where $x \in \Gamma'$ and $f : \Gamma' \to A$ is a continuous function. Choose a (finite) set $\mathcal{Y}$ of coset representatives of Γ'_F in Γ'. There is an isomorphism $\mathcal{U}(A) \to \mathcal{V}(A)$ of Γ'_F-modules which sends a function $f \in \mathcal{U}(A)$ to $\tilde{f}$, where $\tilde{f}(\gamma y) = \gamma^{-1} f(\gamma y)$, $y \in \mathcal{Y}$, $\gamma \in \Gamma'_F$. The fact that $(\mathrm{id} - \gamma'_F) : \mathcal{V}(A) \to \mathcal{V}(A)$ is onto follows from combining 3.9 with the following two observations:

(1) for each n the fixed point set of the action of Γ'_{F_n} on $\mathcal{V}(A)$ is a module over Γ'_F/Γ'_{F_n} induced from a module over the trivial group, and
(2) by Shapiro's lemma, the higher cohomology of a finite group with coefficients in a module induced from the trivial group is zero. □

Remark. The Γ'_F-module isomorphism $\mathcal{U}(A) \to \mathcal{V}(A)$ in the preceding proof might be more familiar in its dual form, which, if G is a discrete group, F a free module over the ring of $\mathbb{Z}[G]$, and A a G-module, gives an isomorphism between $F \otimes_{\mathbb{Z}} A$ provided with the diagonal G-action and $F \otimes_{\mathbb{Z}} A$ provided with the G action induced by the action of G on F.

Recall that $(-)^{\#}$ denotes Pontriagin duality.

6.10 Lemma. *If A is a discrete ℓ-torsion Γ'_F-module, then there is a natural Λ'-module isomorphism $(\mathcal{U}_F(A))^{\#} \cong \Lambda' \otimes_{\Lambda'_F} (A^{\#})$.*

Proof. There is a Γ'_F-module map $\mathcal{U}(A) \to A$ which sends a Γ'_F-equivariant function $f : \Gamma' \to A$ to $f(e)$ (e is the identity element). Taking duals gives a Γ'_F-module map $A^{\#} \to (\mathcal{U}_F(A))^{\#}$ and thus a Λ'-module map $h : \Lambda' \otimes_{\Lambda'_F} (A^{\#}) \to (\mathcal{U}_F(A))^{\#}$. Let $\mathcal{Y}$ be as in the proof of 6.9. The ring Λ' is a free module over Λ'_F with generators from $\mathcal{Y}$. The fact that h is an isomorphism now follows on passing to the dual from the fact that Γ' is free as a Γ'_F-set on the coset representatives $\mathcal{Y}$, so that the map $f \mapsto f|_{\mathcal{Y}}$ gives an isomorphism from $\mathcal{U}(A)$ to a product of copies of A indexed by the set $\mathcal{Y}$. □

6.11 Proposition. *There are natural isomorphisms of Λ'-modules*

$$\hat{\mathcal{K}}^i(K^{\mathrm{red}}R) \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} M(-m) & i = 2m-1 \\ 0 & i \text{ even} \end{cases}$$

Proof. Let X denote $\hat{L}(K^{\mathrm{red}}R_\infty) \wedge \mathcal{N}$. Since X is a torsion spectrum which is a module spectrum over $\hat{\mathcal{K}}$ (5.1), it follows from [10, 6.6] that there are isomorphisms $\hat{\mathcal{K}}_i X = \pi_i \hat{\mathcal{K}} \wedge X \cong \mathcal{U}(\pi_i X)$ for all integers i. (Note that our definition of $\mathcal{U}(-)$, although not identical to the one given by Bousfield, is equivalent to it). Under these isomorphisms the maps $\pi_i(\hat{\mathcal{K}} \wedge X) \to \pi_i X$ induced by the module structure correspond to the maps $\mathcal{U}(\pi_i X) \to \pi_i X$ given by evaluation of functions $\Gamma' \to \pi_i X$ at the identity element of Γ'. By 6.6 and 3.5, this implies that $\hat{\mathcal{K}}_i X$ vanishes for odd i and that for any m, $\hat{\mathcal{K}}_{2m-2} X$ is isomorphic to $\mathcal{U}(M^{\#}(m))$, where $M^{\#}$ is the Pontriagin dual of the Iwasawa module M. We can now compute $\hat{\mathcal{K}}_*(\hat{L}(K^{\mathrm{red}}R) \wedge \mathcal{N})$ using the fibration sequence from 6.6; it is only necessary to compute the map on homotopy induced by $(\mathrm{id} \wedge \gamma'_F) : \hat{\mathcal{K}} \wedge X \to \hat{\mathcal{K}} \wedge X$. By the commutative diagram in 5.1 and naturality of the isomorphisms $\hat{\mathcal{K}}_i X \cong \mathcal{U}(\pi_i X)$, the self map $(\mathrm{id} \wedge \gamma'_F)_*$ of $\pi_i(\hat{\mathcal{K}} \wedge X) = \mathcal{U}(\pi_i X)$ is the unique map of Γ'-modules such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}(\pi_i X) & \xrightarrow{\gamma'_F \cdot (\mathrm{id} \wedge \gamma'_F)_*} & \mathcal{U}(\pi_i X) \\ |_e \downarrow & & |_e \downarrow \\ \pi_i X & \xrightarrow{(\gamma'_F)_*} & \pi_i X \end{array}$$

where $|_e$ is evaluation at the identity $e \in \Gamma'$ and $\gamma'_F \cdot (-)$ refers to multiplication by γ'_F in the Λ'-module $\mathcal{U}(\pi_i X)$. By inspection, then, if $f : \Gamma' \longrightarrow \pi_i X$ is a continuous function and $x \in \Gamma'$, $(\mathrm{id} \wedge \gamma'_F)_* f$ is the function f' with $f'(x) = \gamma'_F f((\gamma'_F)^{-1} x)$, $x \in \Gamma'$. It now follows from a long exact homotopy sequence argument using 3.5, 6.6, and 6.9 that there are isomorphisms

$$\hat{\mathcal{K}}_i(\hat{L}(K^{\mathrm{red}} R) \wedge \mathcal{N}) \cong \begin{cases} \mathcal{U}_F(M^{\#}) & i = -2 \\ 0 & i = -3 \end{cases}$$

The desired formulas are consequences of 4.5, 6.10 and 4.4. □

Proof of 1.7. If R contains μ_ℓ, i.e., $R = R_0$, this follows from 6.11, 4.18, and the splitting formula (5.5) for $\hat{L}(KR_0)$. To jump ahead a bit, the proof of the general case is the same with 9.7 replacing 5.5. Alternatively, one can observe that $\hat{\mathcal{K}}^* \hat{L}(KR) = (\hat{\mathcal{K}}^* \hat{L}(KR_0))^{\Delta_F}$ and compute this Δ_F-fixed submodule by 6.12 below. □

6.12 Remark. The formulas in 1.7 are functorial in a slightly peculiar way. Suppose that α is an automorphism of R or what is the same thing an automorphism of F. Let E be the fixed field of α and α_1 a lift of α to the group $\mathrm{Gal}(F_\infty/E)$, so that by conjugation α_1 acts on the Iwasawa module M. Let α_2 be the image of α_1 under the map $\mathrm{Gal}(F_\infty/E) \longrightarrow \Gamma'_E \subset \Gamma'$ given by the action of $\mathrm{Gal}(F_\infty/E)$ on ℓ-primary roots of unity. Under the isomorphism of 1.7, the action of α on $\hat{\mathcal{K}}^{-1}(KR)$ is composed of the action of α_1 on M and multiplication on the left by $\alpha_2^{-1} \in \Lambda'$. The action of α on $\hat{\mathcal{K}}^0(KR)$ is multiplication by $\alpha_2^{-1} \in \Lambda'$. These statements are proved using the arguments at the end of the proof of 6.11.

There are a few other naturality properties of the isomorphisms from 6.11 which we need. Some of them depend on the fact (§9) that $\Lambda' \otimes_{\Lambda'_{F_n}} M$ is an *excellent* (4.10) Λ'-module.

6.13 Proposition. *The $\hat{L}(K\mathbb{F}_n)$-module structure on $\hat{L}(K^{\mathrm{red}} R_n)$ $(n \geq 0)$ provided by 5.3 corresponds via 4.22(3) to the pair (M, f), where M is the Iwasawa module considered as a module over Λ'_{F_n}, and $f : \Lambda' \otimes_{\Lambda'_{F_n}} M \cong \hat{\mathcal{K}}^{-1} \hat{L}(K^{\mathrm{red}} R_n)$ is the isomorphism given by 6.11.*

Proof. The result follows from a naturality argument using 5.3 and the proof of 6.11. The key point is that the isomorphism between $\hat{\mathcal{K}}^{-1} \hat{L}(K^{\mathrm{red}} R_n)$ and $\Lambda' \otimes_{\Lambda'_{F_n}} M$ from 6.11 depends ultimately upon the choice of $\hat{\mathcal{K}}$-module structure on $\hat{L}(K^{\mathrm{red}} R_\infty) \wedge \mathcal{N}$, and by 5.1 the choice we have made is compatible (up to "weak homotopy") with the $\hat{L}(K\mathbb{F}_n)$-module structure on $\hat{L}(K^{\mathrm{red}} R_n) \wedge \mathcal{N}$.

We will give the bulk of the argument in more detail than the casual reader might want, since there is at least one tricky point. It is enough to work in the case $n = 0$. If A is an ℓ-torsion abelian group, let $\mathcal{U}^{\mathrm{t}}_F(A)$ denote the Γ'-module $\mathcal{U}_F(A^{\mathrm{triv}})$, where A^{triv} is A considered as a trivial Γ'_F-module. The module $\mathcal{U}^{\mathrm{t}}_F(A)$ is just the abelian group of set maps $\Gamma'/\Gamma'_F \longrightarrow A$ with a suitable action of Γ', and by a slight abuse of notation we will denote by $\mathcal{U}^{\mathrm{t}}_F(\mathbb{Z}_\ell)$ the corresponding module of set maps $\Gamma'/\Gamma'_F \longrightarrow \mathbb{Z}_\ell$. By 6.9, for any ℓ-torsion abelian group A there is a short exact sequence

$$0 \longrightarrow \mathcal{U}^{\mathrm{t}}_F(A) \longrightarrow \mathcal{U}(A) \xrightarrow{\mathrm{id}-\gamma'_F} \mathcal{U}(A) \longrightarrow 0\ . \tag{6.14}$$

Suppose now that X is a torsion spectrum with a module spectrum structure map $m : \hat{\mathcal{K}} \wedge X \longrightarrow X$. By 4.19 there is a cofibration sequence

$$\hat{L}(K\mathbb{F}_0) \wedge X \longrightarrow \hat{\mathcal{K}} \wedge X \xrightarrow{\mathrm{id}-(\gamma'_F\wedge\mathrm{id})} \hat{\mathcal{K}} \wedge X\ .$$

The result of Bousfield quoted in the proof of 6.11 gives natural isomorphisms $\alpha : \pi_i\hat{\mathcal{K}} \wedge X \cong \mathcal{U}(\pi_i X)$; combining these isomorphisms with 6.14 and the above cofibration sequence gives natural isomorphisms $\alpha_0 : \pi_i\hat{L}(K\mathbb{F}_0) \wedge X \cong \mathcal{U}^{\mathrm{t}}_F(\pi_i X)$. The module multiplication map m restricts to a module multiplication map $m_0 : \hat{L}(K\mathbb{F}_0) \wedge X \longrightarrow X$. Under the above isomorphisms the map on homotopy groups induced by m_0 sends a function $f \in \mathcal{U}^{\mathrm{t}}_F(\pi_i X)$ to the value of f on the identity coset of Γ'/Γ'_F. It follows easily that for $x \in \pi_i\hat{L}(K\mathbb{F}_n) \wedge X$, $\alpha_0(x)$ is the function $\Gamma'/\Gamma'_F \longrightarrow \pi_i X$ which assigns to $\bar{\gamma} \in \Gamma'/\Gamma'_F$ the image of x under the map on homotopy induced by the composite

$$\hat{L}(K\mathbb{F}_0) \wedge X \xrightarrow{(\gamma)^{-1}\wedge\mathrm{id}} \hat{L}(K\mathbb{F}_0) \wedge X \xrightarrow{m_0} X\ .$$

(Here the action of $(\gamma)^{-1}$ on $\hat{L}(K\mathbb{F}_0)$ is the "internal" one (4.20); it depends up to homotopy only on the image $\bar{\gamma}$ of γ in Γ'/Γ'_F). There is a similar description of the map α.

Let "$\wedge_\ell$" denote the ℓ-completion of the smash product of two spectra. Explicit calculation gives a isomorphism $\alpha'_0 : \pi_0\hat{\mathcal{K}} \wedge_\ell \hat{L}(K\mathbb{F}_0) \longrightarrow \mathcal{U}^{\mathrm{t}}_F(\mathbb{Z}_\ell)$, where for $x \in \pi_0\hat{\mathcal{K}} \wedge_\ell \hat{L}(K\mathbb{F}_0)$ the function $\alpha'_0(x)$ assigns to $\bar{\gamma}$ the image of x under the map on π_0 induced by the composite

$$\hat{\mathcal{K}} \wedge_\ell \hat{L}(K\mathbb{F}_0) \xrightarrow{\mathrm{id}\wedge\gamma^{-1}} \hat{\mathcal{K}} \wedge_\ell \hat{L}(K\mathbb{F}_0) \xrightarrow{m'} \hat{\mathcal{K}}\ .$$

Here m' is derived from the multiplication map for $\hat{\mathcal{K}}$ as a right module spectrum over $\hat{L}(K\mathbb{F}_0)$.

Applying Bousfield's result to $\hat{L}(K\mathbb{F}_0) \wedge X$ gives natural isomorphisms

$$\beta : \pi_i(\hat{\mathcal{K}} \wedge \hat{L}(K\mathbb{F}_0) \wedge X) = \pi_i((\hat{\mathcal{K}} \wedge_\ell \hat{L}(K\mathbb{F}_0)) \wedge X) \xrightarrow{\cong} \mathcal{U}(\mathcal{U}^{\mathrm{t}}_F(\pi_i X))\ .$$

Observe that $\mathcal{U}(\mathcal{U}^{\mathrm{t}}_F(\pi_i X))$ can be identified in a natural way with the space of continuous functions $\Gamma' \times (\Gamma'/\Gamma'_F) \longrightarrow \pi_i X$. Under this identification the map on homotopy groups induced by $(\mathrm{id}\wedge m_0):\hat{\mathcal{K}}\wedge\hat{L}(K\mathbb{F}_0)\wedge X\to\hat{\mathcal{K}}\wedge X$ corresponds by functoriality to the $\mathcal{U}(\mathcal{U}_F^{\mathrm{t}}(\pi_iX))\to\mathcal{U}(\pi_iX)$ induced by the map $\Gamma'\to\Gamma'\times(\Gamma'/\Gamma'_F)$ which sends x to $(x,\bar{e})$. The image under β of an element $x\in\pi_i\hat{\mathcal{K}}\wedge\hat{L}(K\mathbb{F}_0)\wedge X$ is the function $\Gamma'\times(\Gamma'/\Gamma'_F)\to\pi_iX$ which assigns to $(\gamma_1,\bar{\gamma}_2)$ the image of x under the map on homotopy induced by the composite

$$\hat{\mathcal{K}}\wedge\hat{L}(K\mathbb{F}_0)\wedge X\xrightarrow{\gamma_1^{-1}\wedge\gamma_2^{-1}\wedge\mathrm{id}}\hat{\mathcal{K}}\wedge\hat{L}(K\mathbb{F}_0)\wedge X\xrightarrow{\mathrm{id}\wedge m_0}\hat{\mathcal{K}}\wedge X\xrightarrow{m}X\ .$$

Let

$$\beta^{\otimes}:\pi_0(\hat{\mathcal{K}}\wedge_\ell\hat{L}(K\mathbb{F}_0))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}_i(X)\to\mathcal{U}(\mathcal{U}_F^{\mathrm{t}}(\pi_iX))$$

be the composite of β with the $\hat{\mathcal{K}}_*$-Kunneth isomorphism. As above, the image under $\beta^{\otimes}$ of an element $x\otimes y$ is the function $\Gamma'\times(\Gamma'/\Gamma'_F)\to\pi_iX$ which assigns to $(\gamma_1,\bar{\gamma}_2)$ the image of $x\otimes y$ under the composite

$$\begin{aligned}\pi_0(\hat{\mathcal{K}}\wedge_\ell\hat{L}(K\mathbb{F}_0))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}_i(X)&\xrightarrow{(\gamma_2\gamma_1)_*^{-1}\otimes(\gamma_1)_*^{-1}}\pi_0(\hat{\mathcal{K}}\wedge_\ell\hat{L}(K\mathbb{F}_0))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}_i(X)\\&\xrightarrow{\pi_i(m)\hat{\mathcal{K}}_i(m_0)}\pi_iX\end{aligned}.$$

Here we have used the fact that the action of γ_1^{-1} on the "coefficient spectrum" $\hat{\mathcal{K}}$ induces a diagonal action of γ_1^{-1} on the Kunneth factors (4.21). The diagram

$$\begin{array}{ccc}\hat{\mathcal{K}}\wedge\hat{L}(K\mathbb{F}_0)\wedge X & \xrightarrow{\mathrm{id}\wedge m_0} & \hat{\mathcal{K}}\wedge X\\ {\scriptstyle m'\wedge\mathrm{id}}\downarrow & & {\scriptstyle m}\downarrow\\ \hat{\mathcal{K}}\wedge X & \xrightarrow{m} & X\end{array}$$

evidently commutes. Examining the above recipe for $\beta^{\otimes}$ in this light shows that the image under $\beta^{\otimes}$ of $x\otimes y$ is the function which assigns to a pair $(\gamma_1,\bar{\gamma}_2)$ the product of $\alpha_0'(x)(\bar{\gamma}_1\bar{\gamma}_2)$ with $\alpha(y)(\gamma_1)$. Unraveling the constructions leads to the conclusion that the map

$$\begin{aligned}\mathcal{U}_F^{\mathrm{t}}(\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\mathcal{U}(\pi_iX)&\xrightarrow[\cong]{(\alpha_0')^{-1}\otimes\alpha^{-1}}\pi_0(\hat{\mathcal{K}}\wedge_\ell\hat{L}(K\mathbb{F}_0))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}_i(X)\\&\xrightarrow{\hat{\mathcal{K}}_i(m_0)}\hat{\mathcal{K}}_iX\xrightarrow[\cong]{\alpha}\mathcal{U}(\pi_iX)\end{aligned}$$

sends a tensor product $f_1\otimes f_2$ of functions to a function f_3, where $f_3(\gamma)=f_1(\bar{\gamma})f_2(\gamma)$.

One applies these observations to $X=\hat{L}(K^{\mathrm{red}}R_\infty)\wedge\mathcal{N}$ and then uses the fact that the natural map from $Y=\hat{L}(K^{\mathrm{red}}R_0)\wedge\mathcal{N}$ to X is a map of weak module spectra over $\hat{L}(K\mathbb{F}_0)$ which induces an injection on $\hat{\mathcal{K}}_*$ to obtain an explicit formula for the map $\pi_0(\hat{\mathcal{K}}\wedge_\ell\hat{L}(K\mathbb{F}_0))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}_*(Y)\to\hat{\mathcal{K}}_*(Y)$ induced by the module spectrum multiplication. Taking Pontriagin duals then gives the expected formula for the comodule map $\hat{\mathcal{K}}^*(Y)\to\hat{\mathcal{K}}^0(\hat{L}(K\mathbb{F}_0))\otimes_{\mathbb{Z}_\ell}\hat{\mathcal{K}}^*(Y)$. Further details are left to the reader. □

6.15 Remark. By 4.22, the formula $\hat{\mathcal{K}}^{-1}\hat{L}(KR) \cong \Lambda' \otimes_{\Lambda'_F} M$ provides a $\hat{L}(K\mathbb{F})$-module structure on $\hat{L}(K^{\mathrm{red}}R)$. This extends 5.1 to the case $n = -1$.

6.16 Definition. Choose integers n and m, with $-1 \le n \le m$, and let $\{c_i\}$ be a set of coset representatives of Γ'_{F_m} in Γ'_{F_n}. If N is a module over Λ'_{F_n}, the *algebraic transfer map* τ is the Λ'-module map $\Lambda' \otimes_{\Lambda'_{F_n}} N \to \Lambda' \otimes_{\Lambda'_{F_m}} N$ given by the formula

$$\tau(x \otimes x') = \sum_i (c_i)^{-1} x \otimes c_i x' \,.$$

6.17 Proposition. *Let $-1 \le n \le m$ be integers and let $t : \hat{L}(KR_m) \to \hat{L}(KR_n)$ be the K-theory transfer map. Then t induces a natural map $t^{\mathrm{red}} : \hat{L}(K^{\mathrm{red}}R_m) \to \hat{L}(K^{\mathrm{red}}R_n)$. Under the isomorphisms of 6.11, the map on $\hat{\mathcal{K}}^{-1}$ induced by t^{red} is the algebraic transfer τ.*

Proof. By checking the definition of t in terms of module categories, it is not hard to see that there is a (strictly) commutative diagram of spectra

$$\begin{array}{ccc} KR_m & \longrightarrow & K\mathbb{F}_m \\ {\scriptstyle t}\downarrow & & {\scriptstyle t}\downarrow \\ KR_n & \longrightarrow & K\mathbb{F}_n \end{array}$$

in which the right hand map is transfer for the ring extension $\mathbb{F}_n \subset \mathbb{F}_m$; the main fact to check is that there is an isomorphism $\mathbb{F}_n \otimes_{R_n} R_m \cong \mathbb{F}_m$, and this follows from the choice of prime determining $\mathbb{F}_n$. Taking horizontal fibres gives the induced map t^{red}. To compute the map induced by t^{red} on $\hat{\mathcal{K}}^{-1}$ it is enough to compute the map induced by t. Tensoring $R_n \to R_m$ over R_n with R_∞ gives a commutative diagram of rings and an associated commutative transfer diagram

$$(6.18) \qquad \begin{array}{ccc} R_m & \xrightarrow{u} & R_\infty \otimes_{R_n} R_m \\ \uparrow & & {\scriptstyle v}\uparrow \\ R_n & \longrightarrow & R_\infty \end{array} \qquad \begin{array}{ccc} KR_m & \xrightarrow{Ku} & K(R_\infty \otimes_{R_n} R_m) \\ {\scriptstyle t_1}\downarrow & & {\scriptstyle t_2}\downarrow \\ KR_n & \longrightarrow & KR_\infty \end{array} \,.$$

Let $\mathrm{Map}(\Gamma'_{F_n}/\Gamma'_{F_m}, R_\infty)$ be the collection of set maps $\Gamma'_{F_n}/\Gamma'_{F_m} \to R_\infty$; this is a ring under pointwise addition and multiplication. By Galois theory there is a ring isomorphism

$$i : R_\infty \otimes_{R_n} R_m \xrightarrow{\cong} \mathrm{Map}(\Gamma'_{F_n}/\Gamma'_{F_m}, R_\infty)$$

which sends $x \otimes x'$ to the function f given by $f(\gamma) = x\gamma(x')$. This isomorphism is equivariant with respect to the actions of Γ'_{F_n} on the left and $\Gamma'_{F_n}/\Gamma'_{F_m}$ on the right of the objects involved, where these actions are given by the formulas

$$\gamma_1(x \otimes x')\gamma_2 = \gamma_1(x) \otimes \gamma_2(x')$$
$$(\gamma_1 f \gamma_2)(\gamma) = \gamma_1(f(\gamma_2^{-1}\gamma))$$

for $\gamma_1 \in \Gamma'_{F_n}$ and $\gamma_2, \gamma \in \Gamma'_{F_n}/\Gamma'_{F_m}$. After composing with i the map v in 6.18 sends x to the constant function with value x, and the map u sends $x' \in R_m$ to the function f with $f(\gamma) = \gamma(x')$. It is clear that the transfer map t_2 amounts to the direct sum of copies of the identity map of KR_∞. It is now possible to apply the functor $\hat{\mathcal{K}}_{-2}(\hat{L}(-) \wedge \mathcal{N})$ to the transfer diagram in 6.18 and compute the map induced on the right hand column; the key observation is that after applying the functor the horizontal arrows become injections (see the proof of 6.11; observe that $\hat{\mathcal{K}}_{-2}(\hat{L}X \wedge \mathcal{N}) = 0$ for $X = K\mathbb{F}_n$, $X = K\mathbb{F}_m$ or $X = K\mathbb{F}_\infty$, so the distinction between K-theory and reduced K-theory does not matter here). The argument is completed by dualizing as at the end of the proof of 6.11. □

This last proposition has the following curious consequence, which we will use later on. If X and Y are module spectra over a ring spectrum S, let $[X,Y]_S \subset [X,Y]$ denote the set of homotopy classes of maps $X \to Y$ which are maps of S-module spectra.

6.19 Proposition. *Let n and m be integers with $-1 \le n \le m \le \infty$. Then the transfer map $t : \hat{L}(K^{\mathrm{red}}R_m) \to \hat{L}(K^{\mathrm{red}}R_m)$ is a map of $\hat{L}(K\mathbb{F}_n)$-module spectra, and induces a bijection*

$$[\Sigma\hat{L}(K\mathbb{F}_m), \hat{L}(K^{\mathrm{red}}R_m)]_{\hat{L}(K\mathbb{F}_m)} \to [\Sigma\hat{L}(K\mathbb{F}_m), \hat{L}(K^{\mathrm{red}}R_n)]_{\hat{L}(K\mathbb{F}_n)}\,.$$

Proof of 6.19. Let C_i denote the coalgebra $\hat{\mathcal{K}}^0(K\mathbb{F}_i)$. Proving that t is a map of $\hat{L}(K\mathbb{F}_n)$-module spectra amounts (4.15) to observing with the help of 6.17 that $\hat{\mathcal{K}}^1(t)$ is a map of comodules over C_n. To prove the second statement it is enough, again by 4.15 and 6.17, to show that the algebraic transfer τ induces a bijection

$$\mathrm{Hom}_{\Lambda'}(\Lambda' \otimes_{\Lambda'_{F_m}} M, C_m)_{C_m} \to \mathrm{Hom}_{\Lambda'}(\Lambda' \otimes_{\Lambda'_{F_n}} M, C_m)_{C_n}\,,$$

where $\mathrm{Hom}(-,-)_{C_i}$ denotes maps of comodules over C_i. The simplest way to do this is by direct calculation; the bijection does not depend upon any special properties of M. Note that 6.13 gives an explicit description of the comodules in question. □

§7. Finitely generated modules over Λ.

In this section we will describe some results from the structure theory of finitely generated left modules over the ring Λ (this ring is usually called the Iwasawa algebra). This structure theory also applies to modules over Λ_F, since Λ_F is isomorphic to Λ, and with some adjustments to modules over Λ' or Λ'_F, since these rings are isomorphic to direct products of copies of Λ (4.11).

Recall [44, p. 113] [25, p. 124] that choosing a topological generator γ for Γ gives an isomorphism $\Lambda \cong \mathbb{Z}_\ell[[T]]$ sending γ to $1+T$. We will choose such a generator γ, and from now on treat elements of Λ as power series in T.

7.1 Example. Let $u \in \mathbb{Z}_\ell^\times$ denote $c(\gamma)$. If N is the cyclic Λ-module $\Lambda/(f(T))$, then the Tate twisted module $N(-m)$ is isomorphic to $\Lambda/(g(T))$, where $g(T) = f(u^m(1+T)-1)$. (This expression for $g(T)$ does in fact give an element of Λ because u is congruent to 1 mod ℓ.)

7.2 Remark. The isomorphism $\Lambda \cong \mathbb{Z}_\ell[[T]]$ shows that Λ is a regular local ring of global dimension 2. In particular, the functor $\mathrm{Ext}^i_\Lambda(\text{–},\text{–})$ vanishes for $i \geq 3$, and if $N \longrightarrow N'$ is an injection of Λ-modules, the induced map $\mathrm{Ext}^2_\Lambda(N',\text{–}) \longrightarrow \mathrm{Ext}^2_\Lambda(N,\text{–})$ is surjective.

Definition. A polynomial $f(T) \in \Lambda$ is said to be *distinguished* if it is monic and all coefficients except for the leading one are divisible by ℓ. A cyclic module over Λ is said to be *distinguished* if it is free, isomorphic to $\Lambda/(\ell^k)$ for some integer k, or isomorphic to $\Lambda/(f(T))$ for a distinguished polynomial $f(T)$.

7.3 Proposition. [44, p. 271] [25, p. 132]. *Suppose that N is a finitely generated Λ-module. Then there exists some finite direct sum N' of distinguished modules and a map $f : N \longrightarrow N'$ such that* $\ker(f)$ *and* $\mathrm{coker}(f)$ *are finite.*

Note that $\mathbb{Z}/\ell$ has a unique Λ-module structure, namely, the one in which T acts as the zero endomorphism.

7.4 Lemma. *Any finite Λ-module N has a composition series in which the composition factors are isomorphic to $\mathbb{Z}/\ell$.*

Proof. Since T is contained in the maximal ideal of the local ring Λ, by Nakayama's lemma [44, p. 279] T must act on a finite module N as a nilpotent endomorphism. □

7.5 Lemma. *The Λ-module* $\mathrm{Ext}^i_\Lambda(\mathbb{Z}/\ell,\Lambda)$ *is isomorphic to $\mathbb{Z}/\ell$ if $i = 2$ and is zero otherwise. If N is a finite Λ-module, then* $\mathrm{Ext}^i_\Lambda(N,\Lambda) = 0$ *unless $i = 2$,* $\mathrm{Ext}^2_\Lambda(N,\Lambda)$ *is finite, and* $\mathrm{Ext}^2_\Lambda(N,\Lambda) \neq 0$ *if $N \neq 0$.*

7.6 Remark. We will frequently use Λ-modules of the form $\mathrm{Ext}^i_\Lambda(N,\Lambda)$ for a Λ-module N. It is important to use the correct left Λ-module structure on these Ext groups. In its state of nature $\mathrm{Ext}^i_\Lambda(N,\Lambda)$ is a *right* Λ-module, essentially because the left action of Λ on itself is used in forming Ext and it is the right action that survives to give the module structure. We convert $\mathrm{Ext}^i_\Lambda(N,\Lambda)$ into a left Λ-module by using the antiautomorphism of Λ which sends $g \in \Gamma$ to g^{-1}.

Proof of 7.5. The first statement is easy to prove using the short exact sequences

$$\begin{array}{ccccccccc} 0 \to & \Lambda & \xrightarrow{\ell} & \Lambda & \to \Lambda/(\ell) \to 0 \\ 0 \to & \Lambda/(\ell) & \xrightarrow{T} & \Lambda/(\ell) & \to \mathbb{Z}/\ell \to 0 \end{array}.$$

The second follows from 7.4. □

7.7 Proposition. *A finitely generated Λ-module N is excellent (4.10) if and only if N has no finite submodules.*

Proof. Suppose that N has no finite submodules. It is necessary to show that $\mathrm{Ext}^2_\Lambda(N,\text{–})$ is zero. By 7.3 there is an injection $N \longrightarrow N'$, where N' is a direct sum of distinguished modules. By inspection N' has projective dimension ≤ 1, and the desired result follows from 7.2. If on the other hand N contains a finite submodule, then by 7.4 it contains a submodule isomorphic to $\mathbb{Z}/\ell$, and so $\mathrm{Ext}^2_\Lambda(N,\Lambda) \neq 0$ by 7.5 and 7.2. □

7.8 Proposition. *If N is a finitely-generated Λ-module, then its dual* $\operatorname{Hom}_\Lambda(N,\Lambda)$ *is a free Λ-module.*

Proof. We can clearly assume that N has no finite submodules. By 7.3, then, there is an embedding $N \to N'$ with finite cokernel, such that N' is a direct sum of distinguished Λ-modules. It follows from 7.5 that the map $\operatorname{Hom}_\Lambda(N',\Lambda) \to \operatorname{Hom}_\Lambda(N,\Lambda)$ is an isomorphism, and the proof is completed by noticing that since Λ is a domain, the dual of a distinguished module is free. □

7.9 Proposition. *Let N be a finitely generated Λ-module and $N' \subset N$ its maximal finite submodule. Then the restriction map* $\operatorname{Ext}^2_\Lambda(N,\Lambda) \to \operatorname{Ext}^2_\Lambda(N',\Lambda)$ *is an isomorphism.*

Proof. By 7.7 the quotient N/N' is excellent. □

7.10 Proposition. *If N is a finitely generated Λ-module, then* $\operatorname{Ext}^1_\Lambda(N,\Lambda)$ *is a torsion Λ module.*

Proof. This is clear if N is a direct sum of distinguished modules. It follows in the general case from 7.3 and 7.5. □

7.11 Proposition. *Suppose that N is a finitely generated excellent Λ-module. Let $N_i^* = \operatorname{Ext}^i_\Lambda(N,\Lambda)$ for $i = 0$, 1. Then there is an exact sequence of Λ-modules*

$$0 \to \operatorname{Ext}^1_\Lambda(N_1^*,\Lambda) \to N \to \operatorname{Hom}_\Lambda(N_0^*,\Lambda) \to \operatorname{Ext}^2_\Lambda(N_1^*,\Lambda) \to 0\,.$$

Proof. Construct a finitely generated free resolution $\mathcal{R} = (F_1 \to F_0)$ of N, in which each F_i, $i = 0$, 1 is a finitely generated free Λ-module. Let $\mathcal{R}^*$ be the Λ-dual of $\mathcal{R}$, so that $\mathrm{H}^i\,\mathcal{R}^* = N_i^*$, $i = 0$, 1. Since all the modules involved are finitely generated, the double dual $\operatorname{Hom}_\Lambda(\mathcal{R}^*,\Lambda)$ is just the original free resolution, and so has as its only homology group the module N in dimension 0. The exact sequence of the proposition then comes from the standard Universal Coefficient spectral sequence

$$E^2_{i,j} = \operatorname{Ext}^i_\Lambda(\mathrm{H}^j\,\mathcal{R}^*,\Lambda) \Rightarrow \mathrm{H}_{j-i}\operatorname{Hom}_\Lambda(\mathcal{R}^*,\Lambda)$$

which in this special case is very sparse. □

7.12 Example. Let N be the Λ-module $\Lambda/(f(T))$ for a nonzero element $f(T) \in \Lambda$. A calculation with the free resolution

$$\Lambda \xrightarrow{(-)\cdot f(T)} \Lambda \to N$$

shows that $\operatorname{Ext}^i_\Lambda(N,\Lambda)$ is zero for $i \neq 1$ and is isomorphic for $i = 1$ to $N_1^* = \Lambda/(g(T))$, where $g(T) = f((1+T)^{-1} - 1)$ (see 7.6). By 7.11 or another direct calculation, there is a "double duality" isomorphism $N \cong \operatorname{Ext}^1_\Lambda(N_1^*,\Lambda)$.

§8. The spectrum $\hat{L}(K^{\mathrm{red}}R_0)$

In this section we will study the spectrum $\hat{L}(K^{\mathrm{red}}R_0)$, and show that sometimes certain cartesian factors of $\Omega_0^\infty \hat{L}(K^{\mathrm{red}}R_0)$ are also factors of $\Omega_0^\infty \hat{K}^{\mathrm{red}}R_0$. The next section will extend these results to $\hat{L}(K^{\mathrm{red}}R)$. Note that $\Lambda'_{F_0} = \Lambda_{F_0} = \Lambda_F$ (§2), so that 6.11 gives an isomorphism $\hat{\mathcal{K}}^{-1}\hat{L}(K^{\mathrm{red}}R_0) \cong \Lambda' \otimes_{\Lambda_F} M$.

The starting point is the following theorem of Iwasawa.

8.1 Theorem. [23, Thm. 18] *The Iwasawa module M has no finite Λ_F-submodules.*

8.2 Corollary. *The Iwasawa module M is an excellent Λ_F-module, and $\Lambda' \otimes_{\Lambda_F} M$ is an excellent Λ'-module.*

Proof. By [23, Thm. 4], the module M is finitely generated over Λ'_F, so the fact that it is excellent follows from 8.1 and 7.7. The final statement is immediate, since, as a module over Λ_F, Λ' is free on a finite number of generators. □

Proposition 4.17 now gives a determination of $\hat{L}(K^{\mathrm{red}}R_0)$.

8.3 Theorem. *The spectrum $\hat{L}(K^{\mathrm{red}}R_0)$ is of type $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda_F} M, -1)$ or equivalently of type $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda_F} M(-1), 1)$.*

Remark. The two Moore spectra in 8.3 are the same by 4.4. Note that $\Lambda' \otimes_{\Lambda_F} (-)$ commutes with Tate twisting, so the notation $\Lambda' \otimes_{\Lambda_F} M(-1)$ is unambiguous.

Because of 8.3, any algebraic property of the Iwasawa module will be reflected in a geometric property of $\hat{L}(K^{\mathrm{red}}R_0)$; for example a direct sum decomposition of M corresponds to a wedge decomposition of $\hat{L}(K^{\mathrm{red}}R_0)$. In the remainder of this section we will use the property of M described in 8.10 below to prove the following two theorems. These theorems amount to a study of the way in which the Borel classes [7] split off from $\hat{K}R_0$ or $\hat{L}(KR_0)$.

The Λ_F-module A'_∞ is a quotient of A_∞ (§2) and is thus finitely generated by [44, 13.18]. Since Λ_F is noetherian (§7), it follows that A'_∞ has a unique maximal finite submodule.

8.4 Theorem. *Let ℓ^e be the exponent of the maximal finite Λ_F-submodule of A'_∞. Then there are maps $u_{\mathrm{top}} : \hat{L}(K^{\mathrm{red}}R_0) \to (\Sigma\hat{\mathcal{K}})^{r_2(F_0)}$ and $v_{\mathrm{top}} : (\Sigma\hat{\mathcal{K}})^{r_2(F_0)} \to \hat{L}(K^{\mathrm{red}}R_0)$ of module spectra over $\hat{L}(K\mathbb{F}_0)$ such that the composite $u_{\mathrm{top}} \cdot v_{\mathrm{top}}$ is multiplication by ℓ^e. In particular, if A'_∞ contains no finite submodules then $\hat{L}(K^{\mathrm{red}}R_0)$ has $(\Sigma\hat{\mathcal{K}})^{r_2(F_0)}$ as a wedge summand.*

8.5 Theorem. *Let ℓ^e be as in 8.4 and let $\hat{U} = \Omega_0^\infty(\Sigma\hat{\mathcal{K}})$ denote the ℓ-completion of the infinite Unitary group. Then there are maps of spaces $u'_{\mathrm{top}} : \Omega_0^\infty(\hat{K}^{\mathrm{red}}R_0) \to (\hat{U})^{r_2(F_0)}$ and $v'_{\mathrm{top}} : (\hat{U})^{r_2(F_0)} \to \Omega_0^\infty(\hat{K}^{\mathrm{red}}R_0)$ such that the composite $u'_{\mathrm{top}} \cdot v'_{\mathrm{top}}$ is multiplication by ℓ^e with respect to the usual loop space structure on $(\hat{U})^{r_2(F_0)}$. In particular, if A'_∞ has no finite Λ_F-submodules then $(\hat{U})^{r_2(F_0)}$ is a retract of $\Omega_0^\infty(\hat{K}^{\mathrm{red}}R_0)$.*

Remark. In many interesting cases A'_∞ has no finite Λ_F-submodules, but not in all cases (cf. 12.4).

8.6 A Conjecture. In the Introduction of [18] there is a conjecture which in the context of this paper can be strengthened to state that there should be ring spectrum maps maps $h_n^0 : K\mathbb{F}_n \to KR_n$ which fit into commutative diagrams

$$\begin{array}{ccc} K\mathbb{F}_n & \xrightarrow{h_n^0} & KR_n \\ \downarrow & & \downarrow \\ \hat{L}(K\mathbb{F}_n) & \xrightarrow{h_n} & \hat{L}(KR_n) \end{array}$$

with the maps of 5.4. For discussions related to this conjecture see [32]. If we could prove the strengthened conjecture, we could show that the map v_{top} of 8.4 lifts to a map $v^0_{\mathrm{top}} : (P^0\Sigma\hat{\mathcal{K}})^{r_2(F_0)} \to \hat{K}^{\mathrm{red}}R_0$ and consequently in some cases get a splitting of the algebraic K-theory spectrum. As things stand, though, we have to settle in 8.5 for something weaker than a spectrum map v^0_{top}.

Let L_∞ denote $\lim_n \pi_0\hat{L}(K^{\mathrm{red}}R_n)$ where the limit is taken with respect to the norm or transfer map. Note that by 6.4 there is an isomorphism $E'_\infty(\mathrm{red}) \cong \lim_n \pi_1\hat{L}(K^{\mathrm{red}}R_n)$. Passing to the limit with 6.4 and 3.10 shows that there are short exact sequences of Λ'_F-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_\infty & \longrightarrow & L_\infty & \longrightarrow & B_\infty & \longrightarrow & 0 \\ 0 & \longrightarrow & E'_\infty(\mathrm{red}) & \longrightarrow & E'_\infty & \longrightarrow & \lim_n(\mathbb{F}_n^\times)\hat{} & \longrightarrow & 0 \end{array} \quad . \tag{8.7}$$

The Λ'_F-module $\lim_n(\mathbb{F}_n^\times)\hat{}$ in the lower sequence is isomorphic to $\mathbb{Z}_\ell(1)$ and the surjection in the lower sequence is split by the inclusion $\mathbb{Z}_\ell(1) \cong \lim_n \mu(R_n) \to E'_\infty$. As will become clear from the proof of 8.10, both L_∞ and $E'_\infty(\mathrm{red})$ are finitely generated as modules over Λ'_F or equivalently over Λ_F.

8.8 Lemma. *For $n \geq 0$ and $i = 0$, 1 there are natural isomorphisms*

$$\pi_i\hat{L}(K^{\mathrm{red}}R_n) \cong \mathrm{Ext}^{1-i}_{\Lambda_F}(M(-1), \Lambda_F \otimes_{\Lambda_{F_n}} \mathbb{Z}_\ell(0))\,.$$

If $m \geq n$, then under these isomorphisms the transfer map $\pi_i\hat{L}(K^{\mathrm{red}}R_m) \to \pi_i\hat{L}(K^{\mathrm{red}}R_n)$ ($i = 0$, 1) corresponds to the map on Ext^{1-i} induced by the natural surjection $\Lambda_F \otimes_{\Lambda_{F_m}} \mathbb{Z}_\ell(0) \to \Lambda_F \otimes_{\Lambda_{F_n}} \mathbb{Z}_\ell(0)$.

Proof. We will look at the case $i = 1$; the other case is similar. Write $\mathbb{Z}_\ell = \mathbb{Z}_\ell(0)$. There are isomorphisms

$$\begin{aligned} \pi_1\hat{L}(K^{\mathrm{red}}R_n) &\xrightarrow{\cong} \mathrm{Hom}_{\Lambda'}(\Lambda' \otimes_{\Lambda'_{F_n}} M(-1), \mathbb{Z}_\ell) \\ &\cong \mathrm{Hom}_{\Lambda_{F_n}}(M(-1), \mathbb{Z}_\ell) \\ &\cong \mathrm{Hom}_{\Lambda_F}(M(-1), \mathrm{Hom}_{\Lambda_{F_n}}(\Lambda_F, \mathbb{Z}_\ell)) \end{aligned} \tag{8.9}$$

where the first isomorphism comes from 8.3 and 4.15, and the other two from standard adjunctions (note that $\Lambda'_{F_n} = \Lambda_{F_n}$ for $n \geq 0$). By calculation $\mathrm{Hom}_{\Lambda_{F_n}}(\Lambda_F, \mathbb{Z}_\ell)$ is canonically isomorphic as a module over Λ_F to $\Lambda_F \otimes_{\Lambda_{F_n}} \mathbb{Z}_\ell$. (The calculation comes down to observing that for the finite group $G = \Gamma_F/\Gamma_{F_n}$, the natural basis of $\mathbb{Z}_\ell[G]$ given by group elements provides a canonical isomorphism $\mathbb{Z}_\ell[G] \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)$ of $\mathbb{Z}_\ell[G]$-modules.) The statement about the transfer reduces, via the first isomorphism in 8.9, to a calculation with the algebraic transfer (6.17). □

8.10 Theorem. *There is an exact sequence of Λ_F-modules*

$$0 \to \mathrm{Ext}^1_{\Lambda_F}(L_\infty, \Lambda_F) \to M(-1) \to \mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathit{red}), \Lambda_F) \to \mathrm{Ext}^2_{\Lambda_F}(L_\infty, \Lambda_F) \to 0\,.$$

8.11 Remark. This is also an exact sequence of Δ_F-modules if Δ_F acts from the left on the Ext^i groups in the natural way (cf. 7.6); for instance, for $f \in \mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$ and $g \in \Delta_F$, $(g \cdot f)(x) = f(g^{-1}x)$.

Proof of 8.10. Passing to the inverse limit with 8.8 shows that L_∞ (resp. $E'_\infty(\mathrm{red})$) is isomorphic to $\mathrm{Ext}^i_{\Lambda_F}(M(-1), \lim_n(\Lambda_F \otimes_{\Lambda_{F_n}} \mathbb{Z}_\ell))$ for $i = 1$ (resp. $i = 0$). Since $\lim_n(\Lambda_F \otimes_{\Lambda_{F_n}} \mathbb{Z}_\ell)$ is isomorphic to Λ_F, the exact sequence follows from 7.11. □

The first exact sequence of 8.7 describes L_∞ pretty well in terms of familiar objects; our immediate goal now is to get a corresponding description of $E'_\infty(\mathrm{red})$, or equivalently, given the second exact sequence of 8.7, of E'_∞. We will work up to this description, which is provided by 8.17, in stages.

8.12 Lemma. *As in §2, let $E_0(red)$ denote the ℓ-completion of the kernel of the reduction map $\mathcal{O}^\times_{F_0} \to \mathbb{F}^\times_0$. Then* $\mathrm{rank}_{\mathbb{Z}/\ell}(\mathbb{Z}/\ell \otimes E_0(red)) = r_2(F_0) - 1$.

Proof. This is a consequence of the Dirichlet unit theorem. Note that by the choice of $\mathbb{F}_0$, $E_0(\mathrm{red})$ is ℓ-torsion free. □

8.13 Lemma. *The Λ_F-rank of $M(-1)$ is $r_2(F_0)$.*

Proof. Choose a minimal resolution of $M(-1) = \hat{\mathcal{K}}^1\hat{L}(K^{\mathrm{red}}R_0)$ over Λ_F

$$0 \to (\Lambda_F)^a \xrightarrow{C} (\Lambda_F)^b \to M(-1) \to 0 \tag{8.14}$$

and tensor with Λ' to obtain a Λ'-resolution:

$$0 \to (\Lambda')^a \xrightarrow{C} (\Lambda')^b \to \Lambda' \otimes_{\Lambda_F} M(-1)\,.$$

The symbol C denotes a $b \times a$ matrix with coefficients in $\Lambda_F \subset \Lambda'$. It is clear that $\mathrm{rank}_{\Lambda_F} M(-1) = b - a$. As in §4, there is a cofibre sequence

$$\hat{L}(K^{\mathrm{red}}R_0) \to (\Sigma\hat{\mathcal{K}})^{\vee b} \xrightarrow{C^t} (\Sigma\hat{\mathcal{K}})^{\vee a} \tag{8.15}$$

where C^t is a matrix corresponding under the isomorphism $\Lambda' = [\hat{\mathcal{K}}, \hat{\mathcal{K}}]$ to the transpose of C. Since the resolution 8.14 is a minimal one, the elements of C lie in the maximal ideal of Λ' and so C^t induces the zero map on mod ℓ homotopy groups. The cofibration sequence 8.15 therefore shows that the $\mathbb{Z}/\ell$-rank of $\pi_j\hat{L}(K^{\mathrm{red}}R_0) \wedge \mathcal{M}_\ell$ is b if $j = 1$ and a if $j = 0$.

If N is an abelian group write $\mathrm{rank}_{\mathbb{Z}/\ell} N$ for the rank over $\mathbb{Z}/\ell$ of $\mathbb{Z}/\ell \otimes N$, and $\mathrm{corank}_{\mathbb{Z}/\ell} N$ for the $\mathbb{Z}/\ell$-rank of $\ker(\ell : N \to N)$. The universal coefficient theorem for mod ℓ homotopy groups shows that for any appropriately finite spectrum X there are formulas

$$\mathrm{rank}_{\mathbb{Z}/\ell}(\pi_i X \wedge \mathcal{M}_\ell) = \mathrm{rank}_{\mathbb{Z}/\ell}\,\pi_i X + \mathrm{corank}_{\mathbb{Z}/\ell}\,\pi_{i-1} X\,. \tag{8.16}$$

The number of primes inverted in passing from $\mathcal{O}_{F_0}$ to R_0 is s_0, so $\mathrm{rank}_{\mathbb{Z}/\ell} E'_0(\mathrm{red})$ is $s_0 + \mathrm{rank}_{\mathbb{Z}/\ell} E_0(\mathrm{red})$. By 6.4, the proof of 3.10 and 8.12, then, there are equalities

$$\begin{aligned} b &= r_2(F_0) - 1 + s_0 + \mathrm{corank}_{\mathbb{Z}/\ell} A'_0 \\ a &= \mathrm{rank}_{\mathbb{Z}/\ell} A'_0 + s_0 - 1 + \mathrm{corank}_{\mathbb{Z}/\ell}(\pi_{-1}\hat{L}(K^{\mathrm{red}}R_0)) \end{aligned}\;.$$

Now $\mathrm{corank}_{\mathbb{Z}/\ell} A'_0 = \mathrm{rank}_{\mathbb{Z}/\ell} A'_0$ because A'_0 is finite. The group $\pi_{-1}\hat{L}(K^{\mathrm{red}}R_n)$ is a summand of $\pi_{-1}\hat{L}(KR_n) \cong \mathrm{H}^1_{\text{ét}}(R_n; \mathbb{Z}_\ell(0))$ this latter is torsion free because it can be identified with the group of continuous homomorphisms from the étale fundamental group of R_n to $\mathbb{Z}_\ell$ (3.3, [14, II, 2.1]).. It follows immediately that $(b-a) = r_2(F_0)$. □

8.17 Proposition. *There are isomorphisms of Λ_F-modules*

$$\begin{aligned} \mathrm{Hom}_{\Lambda_F}(E'_\infty(red), \Lambda_F) &\cong (\Lambda_F)^{r_2(F_0)} \\ E'_\infty(red) &\cong (\Lambda_F)^{r_2(F_0)} \end{aligned} .$$

Proof. By 8.10 and 7.10 and 7.9, the Λ_F-rank of $\mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$ is the same as the Λ_F-rank of $M(-1)$, which by 8.13 is $r_2(F_0)$. The first isomorphism follows from the fact (7.8) that $\mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$ is free. The second is proved using the first, 8.10, and the isomorphism $E'_\infty(\mathrm{red}) \cong \mathrm{Hom}_{\Lambda_F}(M(-1), \Lambda_F)$ from the proof of 8.10. □

Proof of 8.4. Observe that B_∞ is torsion free and so has no finite submodules (§2, proof of 3.10). By 8.7 and 7.9, ℓ^e is an exponent for $\mathrm{Ext}^2_{\Lambda_F}(L_\infty, \Lambda_F)$. Let N denote the module $\mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$, which by 8.17 is a free Λ_F-module of rank $r_2(F_0)$. By 8.10, there are maps $u_{\mathrm{alg}} : N \to M(-1)$ and $v_{\mathrm{alg}} : M(-1) \to N$ such that the composite $v_{\mathrm{alg}} \cdot u_{\mathrm{alg}}$ is multiplication by ℓ^e. Applying $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda_F} -, 1)$ to these maps gives the maps u_{top} and v_{top}. These are maps of module spectra over $\hat{L}(K\mathbb{F}_0)$ by 4.22 and 6.13. □

The proof of 8.5 depends on a construction of Soulé [41] which was promoted to homotopy theory in [6]. We will use the notation of 6.19. Consider the homomorphism

$$j_n : \pi_1\hat{L}(K^{\mathrm{red}}R_n) \to [\Sigma\hat{L}(K\mathbb{F}_n), \hat{L}(K^{\mathrm{red}}R_0)]_{\hat{L}(K\mathbb{F}_0)}$$

which sends $f : S^1 \to \hat{L}(K^{\mathrm{red}}R_n)$ to the composite

$$\hat{L}(K\mathbb{F}_n) \wedge S^1 \xrightarrow{\mathrm{id}\wedge f} \hat{L}(K\mathbb{F}_n) \wedge \hat{L}(K^{\mathrm{red}}R_n) \xrightarrow{m} \hat{L}(K^{\mathrm{red}}R_n) \xrightarrow{t} \hat{L}(K^{\mathrm{red}}R_0)$$

where m is the module structure map (5.1) and t is the transfer.

8.18 Proposition. *The map j_n is a bijection. For $m > n$ the following diagram commutes*

$$\begin{array}{ccc} \pi_1\hat{L}(K^{\mathrm{red}}R_m) & \xrightarrow{j_m} & [\Sigma\hat{L}(K\mathbb{F}_m), \hat{L}(K^{\mathrm{red}}R_0)]_{\hat{L}(K\mathbb{F}_0)} \\ \downarrow t & & i \downarrow \\ \pi_1\hat{L}(K^{\mathrm{red}}R_n) & \xrightarrow{j_n} & [\Sigma\hat{L}(K\mathbb{F}_n), \hat{L}(K^{\mathrm{red}}R_0)]_{\hat{L}(K\mathbb{F}_0)} \end{array}$$

where the left hand map is the transfer (6.17) and the right hand map is induced by the inclusion $\mathbb{F}_n \to \mathbb{F}_m$.

Proof. The fact that j_n is a bijection comes from combining 6.19 with the fact that the map sending f to $m \cdot (f \wedge \mathrm{id})$ gives a bijection

$$[S^1, \hat{L}(K^{\mathrm{red}}R_n)] \xrightarrow{\cong} [\Sigma\hat{L}(K\mathbb{F}_n), \hat{L}(K^{\mathrm{red}}R_n)]_{\hat{L}(K\mathbb{F}_n)} .$$

The commutativity of the diagram is a consequence of the fact that the transfer is a map of module spectra over $\hat{L}(K\mathbb{F}_n)$ (6.19). $\square$

Since $\hat{\mathcal{K}}^0\hat{\mathcal{K}} \cong \lim_n \hat{\mathcal{K}}^0\hat{L}(K\mathbb{F}_n)$ (4.18), it follows from 4.15 that there is a bijection

$$[\Sigma\hat{\mathcal{K}}, \hat{L}(K^{\mathrm{red}}R_0)]_{\hat{L}(K\mathbb{F}_0)} \xrightarrow{\cong} \lim_n [\Sigma\hat{L}(K\mathbb{F}_n), \hat{L}(K^{\mathrm{red}}R_0)]_{\hat{L}(K\mathbb{F}_0)}\,.$$

Recall that $E'_\infty(\mathrm{red}) = \lim_n \pi_1\hat{L}(K^{\mathrm{red}}R_n)$, and let

$$j_\infty : E'_\infty(\mathrm{red}) \to [\Sigma\hat{\mathcal{K}}, \hat{L}(K^{\mathrm{red}}R_0)]_{\hat{L}(K\mathbb{F}_0)} \tag{8.19}$$

be the homomorphism which sends an element $x = (\dots, x_n, \dots) \in E'_\infty(\mathrm{red})$, where each $x_n \in \pi_1\hat{L}(K^{\mathrm{red}}R_n)$, to the unique map $j_\infty(x)$ which for each n restricts to $j_n(x_n)$ on $\Sigma\hat{L}(K\mathbb{F}_n)$. We have proved the following proposition.

8.20 Proposition. *The map j_∞ displayed in 8.19 is a bijection.*

We can now do something similar to the above with algebraic K-theory, rather than with $\hat{\mathcal{K}}^*$-localized algebraic K-theory. Recall that $\mathrm{B}\mu(R_n)_+$ is the suspension spectrum of the space $\mathrm{B}\mu(\mathbb{F}_n)$. As discussed in §5, for each integer n the spectrum $\hat{K}R_n$ is a module spectrum over $\mathrm{B}\mu(R_n)_+$. Let

$$k'_n : \pi_1\hat{K}^{\mathrm{red}}R_n \to [\Sigma\,\mathrm{B}\mu(R_n)_+, \hat{K}R_0]$$

be the map which assigns to $x \in \pi_1\hat{K}^{\mathrm{red}}R_n$ the composite

$$\mathrm{B}\mu(R_n)_+ \wedge S^1 \xrightarrow{\mathrm{id}\wedge\bar{x}} \mathrm{B}\mu(R_n)_+ \wedge \hat{K}R_n \xrightarrow{m} \hat{K}R_n \xrightarrow{t} \hat{K}R_0$$

where $\bar{x}$ is the image of x in $\pi_1\hat{K}R_n$, m is the module multiplication map and t is the transfer. Note that $\pi_1\hat{K}^{\mathrm{red}}R_n \cong \pi_1\hat{L}(K^{\mathrm{red}}R_n)$ (6.4) so that $E'_\infty(\mathrm{red})$ is isomorphic to the limit $\lim_n \pi_1\hat{K}^{\mathrm{red}}R_n$. Define $k'_\infty : E'_\infty(\mathrm{red}) \to [\Sigma\,\mathrm{B}\mu(R_\infty)_+, \hat{K}R_0]$ to be the homomorphism which assigns to $x = (\dots, x_n, \dots) \in E'_\infty(\mathrm{red})$, where $x_n \in \pi_1\hat{K}^{\mathrm{red}}R_n = \pi_1\hat{L}(K^{\mathrm{red}}R_n)$, the unique map $k'_\infty(x)$ which for each n restricts to $k'_n(x_n)$ on $\Sigma\,\mathrm{B}\mu(R_n)_+$. (The map $k'_\infty(x)$ here is unique because $\hat{K}R_0$ is an ℓ-complete spectrum of finite ℓ-adic type.)

Recall the notation $\hat{\mathrm{B}}G$ and $\hat{\mathrm{B}}G_+$ from 1.13.

8.21 Lemma. *For each $x \in E'_\infty(red)$ there is a map $j'_\infty(x) : \Sigma\hat{\mathrm{B}}S^1_+ \to \hat{K}^{\mathrm{red}}R_0$ such that the following diagram commutes*

$$\begin{array}{ccc} \Sigma\hat{\mathrm{B}}S^1_+ & \xrightarrow{j'_\infty(x)} & \hat{K}^{\mathrm{red}}R_0 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ \Sigma\hat{\mathcal{K}} & \xrightarrow{j_\infty(x)} & \hat{L}(K^{\mathrm{red}}R_0) \end{array} .$$

Here the map i is induced by the usual map $\hat{\mathrm{B}}S^1 = \hat{\mathrm{B}}U(1) \to \hat{\mathrm{B}}U \cong \Omega_0^\infty \hat{\mathcal{K}}$.

Proof. The methods of 4.18 produce a commutative diagram

$$\begin{array}{ccc} \hat{\mathrm{B}}\mu(\mathbb{F}_\infty)_+ & \xrightarrow{\simeq} & \hat{\mathrm{B}}S^1_+ \\ \downarrow & & \downarrow i \\ \hat{L}(K\mathbb{F}_\infty) & \xrightarrow{\simeq} & \hat{\mathcal{K}} \end{array}$$

in which the horizontal arrows, which are equivalences, are given by Brauer lifting. (Note in particular that there is a natural homomorphism $\mu(\mathbb{F}_\infty) \cong \mu(R_\infty) \to \mu(\mathbb{C}) \subset S^1 = \mathrm{U}(1)$.) Use the upper arrow to identify $\hat{\mathrm{B}}\mu(R_\infty)_+$ with $\hat{\mathrm{B}}S^1_+$. For each $x \in E'_\infty(\mathrm{red})$, let $k_\infty(x)$ denote the composite of $j_\infty(x)$ with the map $\hat{L}(K^{\mathrm{red}}R_0) \to \hat{L}(KR_0)$. It follows from the naturality of the above constructions that for each $x \in E'_\infty(\mathrm{red})$ there is a commutative diagram

$$\begin{array}{ccccc} \Sigma\hat{\mathrm{B}}S^1_+ & \xrightarrow{\hat{k}'_\infty(x)} & \hat{K}R_0 & \longrightarrow & \hat{K}\mathbb{F}_0 \\ i\downarrow & & \downarrow & & \downarrow f \\ \Sigma\hat{\mathcal{K}} & \xrightarrow{k_\infty(x)} & \hat{L}(KR_0) & \longrightarrow & \hat{L}(K\mathbb{F}_0) \end{array}$$

in which the lower map is null. Since $\Sigma\hat{\mathrm{B}}S^1_+$ is a 0-connected spectrum and the completion map f is an equivalence on 0-connective covers (4.18), the upper composite is also null and $\hat{k}_\infty(x)'$ lifts to a map $j'_\infty(x)$ with range $\hat{K}^{\mathrm{red}}R_0$. An elementary argument which again uses the fact that $P^0 f$ is an equivalence shows that this lift can be chosen so that the diagram in 8.21 commutes. □

Remark. The above arguments are complicated slightly by the fact that we want to dodge the technical question of whether $\hat{K}^{\mathrm{red}}R_n$ is a module spectrum over $\mathrm{B}\mu(R_n)_+$.

8.22 Lemma. *Let $i : \Sigma\hat{\mathrm{B}}S^1_+ \to \Sigma\hat{\mathcal{K}}$ be the map of 8.21. Then $\Omega_0^\infty i : \Omega_0^\infty(\Sigma\hat{\mathrm{B}}S^1_+) \to \Omega_0^\infty\Sigma\hat{\mathcal{K}} \simeq \hat{U}$ has a right inverse.*

Proof. In this proof all homology and cohomology groups have integral coefficients. We begin by making two observations.

(1) Any graded algebra map $\mathrm{H}^*\,\mathrm{U} \to \mathrm{H}^*\,\mathrm{U}$ which induces a split monomorphism $Q(\mathrm{H}^*\,\mathrm{U}) \to Q(\mathrm{H}^*\,\mathrm{U})$ of multiplicative indecomposables is an isomorphism.

(2) Let $i_0 : \Sigma(\mathrm{B}S^1 \cup \mathrm{pt}) \to \mathrm{U}$ be the map which is adjoint to the usual map $\mathrm{B}S^1 \cup \mathrm{pt} \to \Omega\mathrm{U} \simeq \mathbb{Z} \times \mathrm{BU}$. Then i_0 induces a split monomorphism $Q(\mathrm{H}^*\,\mathrm{U}) \to \mathrm{H}^*(\mathrm{B}S^1 \cup \mathrm{pt})$.

The first observation follows from the fact that $\mathrm{H}^*\,\mathrm{U}$ is finitely generated in each dimension. The second follows from calculation in dimension 1 and in other dimensions $k > 1$ from the fact that, say by the Eilenberg-Moore spectral sequence, the cohomology suspension map $\mathrm{H}^k(\mathrm{SU}) \to \mathrm{H}^k\,\Sigma(\mathrm{BU}) \simeq \mathrm{H}^{k-1}\,\mathrm{BU}$ carries $Q\,\mathrm{H}^k(\mathrm{SU})$ isomorphically onto the group of primitive elements in $\mathrm{H}^{k-1}\,\mathrm{BU}$. These group of primitive elements in turn restricts isomorphically to $\mathrm{H}^k\,\mathrm{B}S^1$, since (dually) the image of $\mathrm{H}_*\,\mathrm{B}S^1$ in $\mathrm{H}_*\,\mathrm{BU}$ generates $\mathrm{H}_*\,\mathrm{BU}$ as a Pontriagin ring.

Let $i' : \Sigma\,\mathrm{B}S^1_+ \to \Sigma\mathcal{K}$ be the natural map which gives rise to i after ℓ-completion. It is enough to prove that $\Omega^\infty_0 i'$ has a right inverse. Consider the diagram

$$\Sigma(\mathrm{B}S^1 \cup \mathrm{pt}) \xrightarrow{i_0} \mathrm{U} \xrightarrow{\kappa} \Omega^\infty_0 \Sigma\,\mathrm{B}S^1_+ \xrightarrow{i'} \mathrm{U},$$

where κ is some map such that $\kappa \cdot i_0$ is adjoint to the identity map of $\Sigma(\mathrm{B}S^1 \cup \mathrm{pt})$ (see [27] or [30, p. 412] for the construction of such a κ). It is clear that the three-fold composite $i' \cdot \kappa \cdot i_0$ is equal to i_0, thus by (2) above that $i' \cdot \kappa$ induces an injection on cohomology indecomposables, and then by (1) above that $i' \cdot \kappa$ is a homotopy equivalence. The map κ gives the desired right inverse to i'. $\square$

Proof of 8.5. By 8.4, 8.20, and 8.21, we can construct a commutative diagram of spectra

$$\begin{array}{ccccc}
(\Sigma\hat{\mathrm{B}}S^1_+)^{r_2(F_0)} & \longrightarrow & \hat{K}^{\mathrm{red}}R_0 & \longrightarrow & (\Sigma\hat{\mathcal{K}})^{r_2(F_0)} \\
{\scriptstyle i^{r_2(F_0)}}\big\downarrow & & \big\downarrow & & {\scriptstyle =}\big\downarrow \\
(\Sigma\hat{\mathcal{K}})^{r_2(F_0)} & \xrightarrow{v_{\mathrm{top}}} & \hat{L}(K^{\mathrm{red}}R_0) & \xrightarrow{u_{\mathrm{top}}} & (\Sigma\hat{\mathcal{K}})^{r_2(F_0)}
\end{array}$$

where i is the map of 8.21. The theorem follows from applying $\Omega^\infty_0(-)$ to this diagram and noticing (8.22) that the map $\Omega^\infty_0(i^{r_2(F_0)})$ has a right inverse. $\square$

§9. Descending to $\hat{L}(K^{\mathrm{red}}R)$.

Our goal here is to extend the results in §8 to the spectrum $\hat{L}(K^{\mathrm{red}}R)$. The fact that the Iwasawa module M is an excellent Λ_F-module (8.2) implies (4.11) that it is an excellent Λ'_F-module, and so by 4.17 the following generalization of 8.3 is immediate.

9.1 Theorem. *The spectrum $\hat{L}(K^{\mathrm{red}}R)$ is of type $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda'_F} M, -1)$ or equivalently of type $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda'_F} M(-1), 1)$.*

The analogues of 8.4 and 8.5 look slightly different. Let $\hat{\mathcal{K}}\mathcal{R}$ denote the ℓ-completion of the periodic real K-theory spectrum. Recall from 6.15 that there is a $\hat{L}(K\mathbb{F})$-module structure on $\hat{L}(K^{\mathrm{red}}R)$.

9.2 Theorem. *Let ℓ^e be the exponent of the maximal finite Λ_F-submodule of A'_∞, and let X be the spectrum $(\Sigma\hat{\mathcal{K}}\mathcal{R})^{r_1(F)} \vee (\Sigma\hat{\mathcal{K}})^{r_2(F)}$. Then there are maps $u_{\mathrm{top}} : \hat{L}(K^{\mathrm{red}}R) \to X$ and $v_{\mathrm{top}} : X \to \hat{L}(K^{\mathrm{red}}R)$ of module spectra over $\hat{L}(K\mathbb{F})$ such that the composite $u_{\mathrm{top}} \cdot v_{\mathrm{top}}$ is multiplication by ℓ^e. In particular, if A'_∞ has no finite Λ_F-submodules then $\hat{L}(K^{\mathrm{red}}R)$ contains X as a wedge summand.*

9.3 Theorem. *Let ℓ^e and X be as in 8.4. Then there are maps $u'_{\mathrm{top}} : \Omega^\infty_0 \hat{K}^{\mathrm{red}}R \to \Omega^\infty_0 X$ and $v'_{\mathrm{top}} : \Omega^\infty_0 X \to \Omega^\infty_0 \hat{K}^{\mathrm{red}}R$ such that the composite $u'_{\mathrm{top}} \cdot v'_{\mathrm{top}}$ is multiplication by ℓ^e with respect to the natural loop space structure on $\Omega^\infty_0 X$. In particular, if A'_∞ has no finite Λ_F-submodules then $\Omega^\infty_0 X$ is a retract of $\Omega^\infty_0 \hat{K}^{\mathrm{red}}R$.*

The proof of these depends in part on studying the relationship between the spectra $\hat{K}^{\mathrm{red}}R_0$ and $\hat{K}^{\mathrm{red}}R$. To begin with we will make a few remarks about certain group actions.

9.4 Splittings arising from group actions. Let G be a subgroup of Δ, which in practice will almost always be Δ_F. It will be very important in what follows that G is finite and of order prime to ℓ. Recall from §2 that the Teichmuller character ω of G is the embedding $G \subset \Gamma' \xrightarrow{c} \mathbb{Z}_\ell^\times$. According to elementary representation theory there are $|G|$ basic idempotents ϵ_i, $i = 0, \dots, |G|-1$ in $\mathbb{Z}_\ell[G]$, where ϵ_i is given by the formula

$$\epsilon_i = (1/|G|) \sum_{g \in G} \omega^{-i}(g) g \,,$$

and any $\mathbb{Z}_\ell[G]$-module N is isomorphic to the direct sum $\oplus_i \epsilon_i N$. If X is an ℓ-complete spectrum with an action of G, then X is a module spectrum over the ring spectrum $(G_+)\hat{}$ and so elements of $\mathbb{Z}_\ell[G] = \pi_0(G_+)\hat{}$ give self-maps of X. In this situation define $\epsilon_i X$ to be the telescope of the sequence $X \xrightarrow{\epsilon_i} X \xrightarrow{\epsilon_i} X \cdots$. There is a map $t_i : \epsilon_i X \to X$ obtained by mapping each term of the telescope to X by ϵ_i. Because the homotopy groups of a telescope are the direct limit of the homotopy groups of its constituents, it is clear that t_i induces an isomorphism $\pi_*(\epsilon_i X) \cong \epsilon_i \pi_* X$ and hence that $\vee t_i : \vee_i \epsilon_i X \to X$ is an equivalence. For each element $g \in G$ there is a diagram

$$\begin{array}{ccccc} X & \xrightarrow{s_i} & \epsilon_i X & \xrightarrow{t_i} & X \\ {\scriptstyle g}\downarrow & & {\scriptstyle \omega^i(g)}\downarrow & & {\scriptstyle \omega^i(g)}\downarrow \\ X & \xrightarrow{s_i} & \epsilon_i X & \xrightarrow{t_i} & X \end{array} \tag{9.5}$$

where s_i is inclusion of the first term of the telescope, so that $s_i t_i = \epsilon_i$, and $\omega^i(g)$ is multiplication by the ℓ-adic unit $\omega^i(g) = (\omega(g))^i$. The large outer square in 9.5 commutes by the definition of ϵ_i and the right hand square trivially commutes. It then follows from the fact that t_i is the inclusion of a wedge summand that the left-hand square in 9.5 commutes, and thus that $\vee s_i : X \to \epsilon_i X$ is an equivariant equivalence, where G acts on each $\epsilon_i X$ by multiplication by the ℓ-adic unit $\omega^i(g)$. It is not hard to show that an equivariant map $X \to Y$ induces equivariant maps $\epsilon_i X \to \epsilon_i Y$ with the appropriate naturality properties; in particular, every equivariant map $\epsilon_i X \to \epsilon_j Y$ for $i \neq j$ is null homotopic.

We will denote $\epsilon_0 X$ by X^G and call it the "homotopy fixed point set" of the action of G on X; note that $\pi_*(X^G) \cong (\pi_* X)^G$.

The *Galois action* of Δ_F on KR_0 is the action derived from the action of Δ_F on R_0 by ring automorphisms.

9.6 Proposition. *The Galois action of Δ_F on KR_0 induces actions of Δ_F on $\hat{K}R_0$, $\hat{K}^{\mathrm{red}}R_0$, $\hat{L}(K^{\mathrm{red}}R_0)$, $K\mathbb{F}_0$, and $\hat{L}(K\mathbb{F}_0)$. With these actions there are natural homotopy equivalences $\hat{K}R \simeq (\hat{K}R_0)^{\Delta_F}$, $\hat{K}^{\mathrm{red}}R \simeq (\hat{K}^{\mathrm{red}}R_0)^{\Delta_F}$, $\hat{L}(K^{\mathrm{red}}R) \simeq \hat{L}(K^{\mathrm{red}}R_0)^{\Delta_F}$, $K\mathbb{F} \simeq (K\mathbb{F}_0)^{\Delta_F}$ and $\hat{L}(K\mathbb{F}) \simeq \hat{L}(K\mathbb{F}_0)^{\Delta_F}$.*

9.7 Example. Applying $(-)^{\Delta_F}$ to the composite

$$\hat{L}(K\mathbb{F}_0) \xrightarrow{h_0} \hat{L}(KR_0) \xrightarrow{\hat{L}(\pi_0)} \hat{L}(K\mathbb{F}_0)$$

from 5.4 gives both a ring spectrum map $\hat{L}(K\mathbb{F}) \to \hat{L}(KR)$ and a wedge decomposition $\hat{L}(KR) \simeq \hat{L}(K\mathbb{F}) \vee \hat{L}(K^{\mathrm{red}}R)$.

Proof of 9.6. The group Δ_F acts on $\hat{K}R_0$ and $\hat{K}\mathbb{F}_0$ by functoriality. The natural map $\hat{K}R \to \hat{K}R_0$ is equivariant with respect to the trivial action of Δ_F on $\hat{K}R$, and the induced map $\hat{K}R \simeq (\hat{K}R)^{\Delta_F} \to (\hat{K}R_0)^{\Delta_F}$ is an equivalence by a standard argument using homotopy group calculations (9.4) and the transfer. (In brief, let $f : \hat{K}R \to \hat{K}R_0$ be the usual map and $t : \hat{K}R_0 \to \hat{K}R$ the transfer [35, p. 103]. A calculation shows that $f \cdot t$ is $|\Delta_F|\epsilon_0$ and that $t \cdot f$ is multiplication by the element $[R_0]$ in K_0R represented by the R-module R_0. It is straightforward to see that $[R_0]$ differs from $[R^{d_F}]$ by a nilpotent element of K_0R, and thus becomes invertible in $\pi_0\hat{K}R$ because $d_F = |\Delta_F|$ is relatively prime to ℓ. The desired result follows easily.) For similar reasons the map $\hat{K}\mathbb{F} \to (\hat{K}\mathbb{F}_0)^{\Delta_F}$ is an equivalence. The map $\hat{K}R_0 \to \hat{K}\mathbb{F}_0$ is Δ_F-equivariant and so induces maps $\epsilon_i\hat{K}R_0 \to \epsilon_i\hat{K}\mathbb{F}_0$ with fibres, say X_i. The action of $g \in \Delta_F$ on $\epsilon_i\hat{K}R_0$ or $\epsilon_i\hat{K}\mathbb{F}_0$ is multiplication by the ℓ-adic integer $\omega^i(g)$, and so these maps easily lift to an action of Δ_F on X_i. Taking wedges gives an action of Δ_F on $\vee_i X_i = \hat{K}^{\mathrm{red}}R_0$, with respect to which $(\hat{K}^{\mathrm{red}}R_0)^{\Delta_F}$ is $X_0 = \hat{K}^{\mathrm{red}}R$. The equivalence $\hat{L}(K^{\mathrm{red}}R) \simeq \hat{L}(K^{\mathrm{red}}R_0)^{\Delta_F}$ and the corresponding finite field statement follow from the fact that $\hat{L}(-)$ (like any localization functor) preserves finite wedges, and the fact that multiplication by an ℓ-adic unit on a spectrum X induces multiplication by the same unit on $\hat{L}X$ (this property can be derived from the description of $\hat{L}X$ in 5.10). □

In order to exploit 9.6, we need a Δ_F-equivariant description of $E'_\infty(\mathrm{red})$ analogous to 8.17. We will gradually work our way up to giving this description in 9.10 below. The next proposition is a refinement of 8.12.

9.8 Lemma. *For $0 \le i \le d_F - 1 = \#\Delta_F - 1$ there are equalities*

$$\mathrm{rank}_{\mathbb{Z}/\ell}\, \epsilon_i(\mathbb{Z}/\ell \otimes E_0(\mathit{red})) = \begin{cases} r_1(F) + r_2(F) - 1 & i = 0 \\ r_1(F) + r_2(F) & i \text{ even, } i \neq 0 \\ r_2(F) & i \text{ odd} \end{cases}.$$

Proof. Let U be the quotient of $\mathcal{O}^\times_{F_0}$ by its torsion subgroup. By choice of $\mathbb{F}_0$ there is a natural Δ_F-isomorphism $\mathbb{Z}/\ell \otimes U \cong \mathbb{Z}/\ell \otimes E_0(\mathrm{red})$, so we can prove the lemma by working with U. Let X denote the set of equivalence classes under complex conjugation of embeddings of F_0 in $\mathbb{C}$. (Note that there are no embeddings of F_0 in $\mathbb{R}$.) The usual proof of the Dirichlet unit theorem produces a short exact sequence

$$0 \to \mathbb{R} \otimes_{\mathbb{Z}} U \to (\mathbb{R})^X \xrightarrow{a} \mathbb{R} \to 0$$

in which a is the obvious sum map. This exact sequence is Δ_F-equivariant, where Δ_F acts on the middle group by permuting embeddings among themselves and Δ_F acts trivially on the right-hand group. This makes it possible to use the exact sequence to compute the character of the action of Δ_F on $\mathbb{R} \otimes_{\mathbb{Z}} U$, or equivalently the character of the action of Δ_F on $\mathbb{C} \otimes_{\mathbb{Z}} U$ or on $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} U$. Since U is torsion free, $\mathrm{rank}_{Ql}\, \epsilon_i(\mathbb{Q}_\ell \otimes_{\mathbb{Z}} U) = \mathrm{rank}_{\mathbb{Z}/\ell}\, \epsilon_i(\mathbb{Z}/\ell \otimes U)$. The lemma now follows from a straightforward calculation; the important point is that the embeddings of F in $\mathbb{C}$ correspond to the orbits of Δ_F on X and that such an embedding is real if and only if the isotropy subgroup of the corresponding orbit in X contains an element of order 2. □

9.9 Lemma. *For $0 \le i \le d_F - 1 = \#\Delta_F - 1$ there are equalities*

$$\operatorname{rank}_{\Lambda_F} \epsilon_i(M(-1)) = \begin{cases} r_1(F) + r_2(F) & i \text{ even} \\ r_2(F) & i \text{ odd} \end{cases}.$$

Proof. Choose a projective resolution $\mathcal{R} = (N_1 \longrightarrow N_0)$ of M over Λ'_F so that for each idempotent ϵ_i, $\epsilon_i\mathcal{R}$ is a minimal resolution of $\epsilon_i M$ over Λ_F. Let $a_i = \operatorname{rank}_{\Lambda_F} \epsilon_i N_1$ and $b_i = \operatorname{rank}_{\Lambda_F} \epsilon_i N_0$. Then $\operatorname{rank}_{\Lambda_F} \epsilon_i M = b_i - a_i$, and for reasons discussed in the proof of 8.13 the $\mathbb{Z}/\ell$-rank of $\epsilon_i\pi_j\hat{L}(K^{\text{red}}R_0)\wedge\mathcal{M}_\ell$ is b_i if $j=1$ and a_i if $j=0$. Recall from 6.4 that $\pi_1\hat{L}(K^{\text{red}}R_0)$ is $E'_0(\text{red})$. The long exact K-theory sequence associated to the localization formula $R_0 = \mathcal{O}_{F_0}[1/\ell]$ therefore gives in low dimensions an exact sequence of Δ_F-modules

$$0 \longrightarrow \mathbb{Z}_\ell \otimes E_0(\text{red}) \longrightarrow (\pi_1\hat{L}(K^{\text{red}}R_0) = E'_0(\text{red})) \longrightarrow \mathbb{Z}_\ell[S_0] \longrightarrow A_0 \longrightarrow A'_0 \longrightarrow 0\,.$$

By 6.4 and 3.10 there is an exact sequence of Δ_F-modules

$$0 \longrightarrow A'_0 \longrightarrow \pi_0\hat{L}(K^{\text{red}}R_0) \longrightarrow \mathbb{Z}_\ell[S_0] \longrightarrow \mathbb{Z}_\ell(0) \longrightarrow 0\,.$$

(Recall from §2 that S_0 is the set of primes above ℓ in R_0.) If N is a $\mathbb{Z}_\ell$-module with an action of Δ_F, write $\operatorname{rank}^i_{\mathbb{Z}/\ell} N$ (resp. $\operatorname{corank}^i_{\mathbb{Z}/\ell} N$) for the $\mathbb{Z}/\ell$-rank (resp. corank) of $\epsilon_i N$ (see the proof of 8.13). Since the class group A_0 is finite and $\epsilon_i\mathbb{Z}_\ell[S_0]$ is torsion-free for each i, the first exact sequence above gives

$$\operatorname{rank}^i_{\mathbb{Z}/\ell}(\pi_1\hat{L}(K^{\text{red}}R_0)) = \operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell \otimes E_0(\text{red})) + \operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell[S_0])\ .$$

The second exact sequence above gives equalities

$$\begin{aligned}\operatorname{corank}^i_{\mathbb{Z}/\ell}(\pi_0\hat{L}(K^{\text{red}}R_0)) &= \operatorname{corank}^i_{\mathbb{Z}/\ell}(A'_0)\\ \operatorname{rank}^i_{\mathbb{Z}/\ell}(\pi_0\hat{L}(K^{\text{red}}R_0)) &= \operatorname{rank}^i_{\mathbb{Z}/\ell}(A'_0) + \operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell[S_0]) - \delta^i_0\end{aligned}$$

where δ^i_0 is Kronecker delta. Formula 8.16 gives equalities

$$\begin{aligned}b_i &= \operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell \otimes E_0(\text{red})) + \operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell[S_0]) + \operatorname{corank}^i_{\mathbb{Z}/\ell}(A'_0)\\ a_i &= \operatorname{rank}^i_{\mathbb{Z}/\ell}(A'_0) + \operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell[S_0]) - \delta^i_0\end{aligned}$$

where we refer to the remark at the end of the proof of 8.13 for the fact that $\operatorname{corank}^i_{\mathbb{Z}/\ell}(\pi_{-1}\hat{L}(KR_0))$ does not appear on the second line. Lemma 9.8determines the number $\operatorname{rank}^i_{\mathbb{Z}/\ell}(\mathbb{Z}_\ell\otimes E_0(\text{red}))$. Since A'_0 is finite, $\operatorname{corank}^i_{\mathbb{Z}/\ell}(A'_0) = \operatorname{rank}^i_{\mathbb{Z}/\ell}(A'_0)$. The proof is completed by computing $b_i - a_i$. □

Recall (§2) that $\Lambda'_F \cong \Lambda_F[\Delta_F]$. Suppose that Δ_F contains an element t of order 2 (necessarily unique). Given a module N over Λ_F, let N_{ev} (resp. N_{odd}) denote the submodule of N on which t acts trivially (resp. acts by multiplication by (-1)). If Δ_F has odd order, set $N_{\text{ev}} = 0$ and $N_{\text{odd}} = N$. Let $\Lambda^+_F = (\Lambda'_F)_{\text{ev}}$.

9.10 Proposition. *There are isomorphisms of Λ'_F-modules*

$$\begin{aligned} \mathrm{Hom}_{\Lambda_F}(E'_\infty(red), \Lambda_F) &\cong (\Lambda_F^+)^{r_1(F)} \oplus (\Lambda'_F)^{r_2(F)} \\ E'_\infty(red) &\cong (\Lambda_F^+)^{r_1(F)} \oplus (\Lambda'_F)^{r_2(F)} . \end{aligned}$$

Proof. Given the above two lemmas, this is very similar to the proof of 8.17. □

The following proposition results from identifying the action of complex conjugation on $\hat{\mathcal{K}}$ with the action of the element of order 2 in $\Delta \subset \Gamma'$ on $\hat{\mathcal{K}}$. Let $\{\pm 1\} \subset \Delta$ denote the subgroup of order 2.

9.11 Proposition. *The spectrum $\hat{\mathcal{K}}\mathcal{R}$ is the wedge summand of $\hat{\mathcal{K}}$ given by $(\hat{\mathcal{K}})^{\{\pm 1\}}$ (see 9.4). Consequently, if Δ_F contains an element of order 2 there is an equivalence $\hat{\mathcal{K}}\mathcal{R} \simeq \mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda'_F} \Lambda_F^+, 0)$.*

Proof of 9.2. By 8.7 and 7.9, ℓ^e is an exponent for $\mathrm{Ext}^2_{\Lambda_F}(L_\infty, \Lambda_F)$. Let N denote the Λ'_F-module $\mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$ described in 9.10. Since N is a projective Λ'_F-module, it follows from 8.10 that there are Λ'_F-module maps $u_{\mathrm{alg}} : N \to M(-1)$ and $v_{\mathrm{alg}} : M(-1) \to N$ such that $v_{\mathrm{alg}} \cdot u_{\mathrm{alg}}$ is multiplication by ℓ^e. Applying $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda'_F} -, 1)$ to these maps (and using 9.11 if $|\Delta_F|$ is even) finishes the proof; the module spectrum statement comes from 4.22 and 6.13 □

The discussion preceding 8.20 can be combined with a naturality argument to give the following proposition.

9.12 Proposition. *The transfer construction of §8 gives a bijection*

$$J_\infty : E'_\infty(red) \xrightarrow{\cong} [\Sigma\hat{\mathcal{K}}, \hat{L}(K^{\mathrm{red}}R)]_{\hat{L}(K\mathbb{F})} .$$

This bijection is Γ'_F-equivariant, in the sense that if $\gamma \in \Gamma'_F$ and $x \in E'_\infty(red)$, then $J_\infty(\gamma x) = J_\infty(x) \cdot \gamma^{-1}$.

Remark. The action of Γ'_F on $\hat{\mathcal{K}}$ implicit in 9.12 is of course the one that results from the canonical embedding $\Gamma'_F \to \Gamma'$ (§4).

Proof of 9.3. We will only sketch the argument, since it is very similar to the arguments at the end of §8. Suppose that Δ_F has an element of order 2; the other case is simpler. The map $\Lambda' \otimes_{\Lambda'_F} M(-1) \to \Lambda' \otimes_{\Lambda'_F} \mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$ obtained from 8.10 gives by 9.10 a set of maps $x_i : \Lambda' \otimes_{\Lambda'_F} M(-1) \to \hat{\mathcal{K}}^0\hat{\mathcal{K}}\mathcal{R}$ $(i = 1, \dots, r_1(F))$ and a set of maps $y_j : \Lambda' \otimes_{\Lambda'_F} M(-1) \to \hat{\mathcal{K}}^*\hat{\mathcal{K}}$ $(j = 1, \dots, r_2(F))$. Let $x'_i : \Lambda' \otimes_{\Lambda'_F} M(-1) \to \hat{\mathcal{K}}^0\hat{\mathcal{K}}$ be the composite of x_i with the summand (9.11) inclusion $\hat{\mathcal{K}}^0\hat{\mathcal{K}}\mathcal{R} \to \hat{\mathcal{K}}^0\hat{\mathcal{K}}$. Each map x'_i is equivariant with respect to the subgroup $\{\pm 1\} \subset \Delta$, where $\{\pm 1\}$ acts on $\Lambda' \otimes_{\Lambda'_F} M(-1)$ trivially and on $\hat{\mathcal{K}}^0(\hat{\mathcal{K}})$ via the natural action on the "inner" $\hat{\mathcal{K}}$. Denote the maps $\hat{L}(K^{\mathrm{red}}R) \to \hat{\mathcal{K}}$ corresponding (4.15) to x'_i and y_j again by the symbols x'_i and y_j. Each map $x'_i : \hat{L}(K^{\mathrm{red}}R) \to \hat{\mathcal{K}}$ is again equivariant with respect to $\{\pm 1\}$, where $\{\pm 1\}$ acts trivially on $\hat{L}(K^{\mathrm{red}}R)$ and in the natural way on $\hat{\mathcal{K}}$. The maps x'_i and y_j are maps of module spectra over $\hat{L}(K\mathbb{F})$ by 4.22. By 9.12, each map y_j corresponds to an element b_i of $E'_\infty(\mathrm{red})$, and each map x_i' to an element $a_i \in (E'_\infty(\text{red}))^{\{\pm 1\}}$. The construction in the proof of 8.21 shows that for each a_i, say, there is a map $J'_\infty(a_i) : \Sigma\hat{\mathrm{B}}S^1_+ \to \hat{K}^{\text{red}}R$ which is $\{\pm 1\}$ equivariant (with respect to the trivial action of $\{\pm 1\}$ on $\hat{K}^{\text{red}}R$ and the complex conjugation action of $\{\pm 1\}$ on $\hat{\mathrm{B}}S^1_+$; as in 8.21 these maps fit into commutative diagrams

$$\begin{array}{ccc} \Sigma\hat{\mathrm{B}}S^1_+ & \xrightarrow{J'_\infty(a_i)} & \hat{K}^{\text{red}}R \\ \downarrow & & \downarrow \\ \Sigma\hat{\mathcal{K}} & \xrightarrow{x_i'=J_\infty(a_i)} & \hat{L}(K^{\text{red}}R) \end{array}$$

Taking "homotopy fixed points" (9.4) with respect to $\{\pm 1\}$ gives commutative diagrams

$$\begin{array}{ccc} (\Sigma\hat{\mathrm{B}}S^1_+)^{\{\pm 1\}} & \longrightarrow & (\hat{K}^{\text{red}}R)^{\{\pm 1\}} = \hat{K}^{\text{red}}R \\ \downarrow & & \downarrow \\ (\Sigma\hat{\mathcal{K}})^{\{\pm 1\}} = \hat{\mathcal{K}}\mathcal{R} & \longrightarrow & \hat{L}(K^{\text{red}}R)^{\{\pm 1\}} = \hat{L}(K^{\text{red}}R) \end{array}$$

There are similar maps $J'_\infty(b_j) : \Sigma\hat{\mathrm{B}}S^1_+ \to \hat{K}^{\text{red}}R$, but no action of $\{\pm 1\}$ to take into account in this case. For notational simplicity let $Y = (\hat{\mathrm{B}}S^1_+)^{\{\pm 1\}}$. The argument is completed by assembling all of the maps into a diagram of spectra

$$\begin{array}{ccccc} (\Sigma Y)^{r_1(F)} \vee (\Sigma\hat{\mathrm{B}}S^1_+)^{r_2(F)} & \longrightarrow & \hat{K}^{\text{red}}R & \longrightarrow & (\Sigma\hat{\mathcal{K}}\mathcal{R})^{r_1(F)} \vee (\Sigma\hat{\mathcal{K}})^{r_2(F)} \\ \downarrow & & \downarrow & & \downarrow = \\ (\Sigma\hat{\mathcal{K}}\mathcal{R})^{r_1(F)} \vee (\Sigma\hat{\mathcal{K}})^{r_2(F)} & \xrightarrow{v_{\text{top}}} & \hat{L}(K^{\text{red}}R) & \xrightarrow{u_{\text{top}}} & (\Sigma\hat{\mathcal{K}}\mathcal{R})^{r_1(F)} \vee (\Sigma\hat{\mathcal{K}})^{r_2(F)} \end{array}$$

and arguing from 8.22 that the vertical arrow on the left gains a right inverse when the functor $\Omega_0^\infty(-)$ is applied. □

§10. Homology calculations

The Lichtenbaum-Quillen conjecture for R is equivalent to the conjecture that the natural map $\mathrm{BGL}(R) \to \Omega_0^\infty \hat{L}(KR)$ is an isomorphism on $\mathrm{H}^* = \mathrm{H}^*(-;\mathbb{Z}/\ell)$. Since this conjecture is true for a finite field (cf. proof of 4.18), the splitting described in 9.7 leads to the following result.

10.1 Proposition. *If the Lichtenbaum-Quillen conjecture is true for R, then there is an isomorphism of Hopf algebras*

$$\mathrm{H}^*\,\mathrm{BGL}(R) \cong \mathrm{H}^*\,\Omega_0^\infty \hat{L}(K^{\text{red}}R) \otimes \mathrm{H}^*\,\mathrm{BGL}(\mathbb{F})\,. \tag{10.2}$$

Quillen computed $\mathrm{H}^*\,\mathrm{BGL}(\mathbb{F})$ in [34]. In this section we will concentrate on computing the other factor on the right hand side of 10.2. This gives an explicit conjectural calculation of $\mathrm{H}^*\,\mathrm{BGL}(R)$ which extends similar conjectural calculations from [16] and [17].

10.3 The case of R_0. We will treat this case to begin with; it is a little simpler than the general one. If X and Y are spectra, write $X \wedge_\ell Y$ for the ℓ-completion of

$X \wedge Y$. Let i be either 0 or 1, and let N be a finitely generated $\mathbb{Z}_\ell$-module which is free if $i = 1$; recall that $\mathcal{M}(N, i)$ is a Moore spectrum of type (N, i). Define the Hopf algebra $H(N, i)$ by the formula

$$H(N, i) = \mathrm{H}^* \, \Omega_0^\infty(\mathcal{M}(N, i) \wedge_\ell \hat{\mathcal{K}}) \, .$$

10.4 Theorem. *There is an isomorphism of Hopf algebras*

$$\mathrm{H}^* \, \Omega_0^\infty \hat{L}(K^{\mathrm{red}} R_0) \cong H(E_0'(\mathit{red}), 1) \otimes H(A_0', 0) \otimes H(B_0, 0) \, .$$

Remark. The homology tensor product formula in 10.4 and the corresponding formula in 10.11 below do not usually reflect cartesian product decompositions of the spaces in question.

10.5 Remark. The Hopf algebras $H(N, i)$ can be described very explicitly. The algebra $H(N, 1)$ is $\mathrm{H}^*(\mathrm{U}^r)$, where U is the infinite Unitary group and r is the $\mathbb{Z}_\ell$-rank of N (recall that we restrict to free $\mathbb{Z}_\ell$-modules N in considering $H(N, 1)$). For the case $i = 0$, write N as a sum of $(\mathbb{Z}_\ell)^r$ and various cyclic groups $\mathbb{Z}/\ell^{k_j}$. Then $H(N, 0)$ is the tensor product of $\mathrm{H}^*(\mathrm{BU}^r)$ with the algebras $H(\mathbb{Z}/\ell^{k_j}, 0)$. For any integer k, $H(\mathbb{Z}/\ell^k, 0)$ is the cohomology of the first delooping of the fibre of the ℓ^k'th power map $\mathrm{BU} \to \mathrm{BU}$. To understand this algebra, let $P(k)$ be the Hopf algebra dual to $(\mathrm{H}_* \, \mathrm{BU})/I_k$, where $I_k \subset \mathrm{H}_* \, \mathrm{BU}$ is the ideal generated by elements of the form x^{ℓ^k}, $|x| > 0$. In the notation of [33, §3], $P(k)$ is isomorphic to the tensor product $\otimes_{(n,\ell)=1} A_{n,k-1}$, where $A_{n,k-1}$ is a certain polynomial algebra on generators $a_{n,i}$ of degree $2n\ell^i$, $0 \le i \le k$. Let $P'(k)$ denote the exterior algebra which is the "delooping" of $P(k)$, so that $P'(k)$ is an exterior algebra on (primitive) generators $\sigma_{n,i}$ of degree $2n\ell^i + 1$, where $(n, \ell) = 1$ and $0 \le i < k$. Then $H(\mathbb{Z}/\ell^k, 1) = P(k) \otimes P'(k)$.

10.6 Remark. There are alternate ways to present 10.4 which exploit the isomorphism $H(N \times N', i) \cong H(N, i) \otimes H(N', i)$. For example, $E_0'(\mathrm{red})$ is isomorphic as a $\mathbb{Z}_\ell$-module to $(\mathbb{Z}_\ell)^{r_2(F_0)+s_0-1}$ (see the proof of 8.13), and B_0 is isomorphic to $(\mathbb{Z}_\ell)^{s_0-1}$. This gives

$$\mathrm{H}^* \, \Omega_0^\infty \hat{L}(K^{\mathrm{red}} R_0) \cong \mathrm{H}^*(\mathrm{U}^{r_2(F_0)+s_0-1}) \otimes \mathrm{H}^*(\mathrm{BU}^{s_0-1}) \otimes H(A_0', 0) \, .$$

For the next two lemmas, suppose that X is a spectrum which lies in a cofibration sequence

$$(\hat{\mathcal{K}})^b \xrightarrow{C} (\hat{\mathcal{K}})^a \to X \, . \tag{10.7}$$

where C is an $a \times b$ matrix with coefficients in $\Lambda \cong \mathbb{Z}_\ell[[T]]$. Write $C(0)$ for the $a \times b$ matrix over $\mathbb{Z}_\ell$ obtained by applying to C the augmentation homomorphism $\Lambda \to \mathbb{Z}_\ell$.

10.8 Lemma. *Define $\mathbb{Z}_\ell$-modules U_0 and U_1 by the exact sequence*

$$0 \longrightarrow U_1 \longrightarrow (\mathbb{Z}_\ell)^b \xrightarrow{C(0)} (\mathbb{Z}_\ell)^a \longrightarrow U_0 \longrightarrow 0\,.$$

Then $U_i \cong \pi_i X$, $i = 0$, 1. If $C = C(0)$, i.e, if C is itself a matrix over $\mathbb{Z}_\ell$, then

$$X \simeq \mathcal{M}(U_1, 1) \wedge_\ell \hat{\mathcal{K}} \bigvee \mathcal{M}(U_0, 0) \wedge_\ell \hat{\mathcal{K}}\,.$$

Proof. The first statement results from the fact that $\pi_0\hat{\mathcal{K}}$ is isomorphic to $\mathbb{Z}_\ell(0)$ as a module over Λ, so that $C(0)$ gives the homomorphism $\pi_0\hat{\mathcal{K}}^a \longrightarrow \pi_0\hat{\mathcal{K}}^b$ induced by C. The second statement is obvious. □

10.9 Lemma. *The Hopf algebra $\mathrm{H}_*\,\Omega_0^\infty X$ depends only on $C(0)$.*

Proof. Applying $\Omega_0^\infty(-)$ to 10.7 gives a fibre sequence

$$(F \simeq \pi_1 X \times (\hat{\mathrm{B}}\mathrm{U})^b) \longrightarrow (\hat{\mathrm{B}}\mathrm{U})^a \longrightarrow \Omega_0^\infty X \tag{10.10}$$

where $\pi_1 X$ is the finitely generated free $\mathbb{Z}_\ell$-module denoted U_1 in Lemma 10.8. To compute $\mathrm{H}_*\,\Omega_0^\infty X$ we can use the Rothenberg-Steenrod spectral sequence. Let $\bar{C}$ denote the induced map $\otimes^b\,\mathrm{H}_*\,\hat{\mathrm{B}}\mathrm{U} \longrightarrow \otimes^a\,\mathrm{H}_*\,\hat{\mathrm{B}}\mathrm{U}$. The category of bicommutative Hopf algebras is abelian, and so $\bar{C}$ is an $a \times b$ matrix over the ring $\mathcal{E}$ of Hopf algebra endomorphisms of $\mathrm{H}_*\,\hat{\mathrm{B}}\mathrm{U}$. However, as noted in [33, 3.9], the endomorphism of $\mathrm{H}_*\,\hat{\mathrm{B}}\mathrm{U}$ induced by a power series (§7) in Λ is just the Hopf power given by $f(0) \in \mathbb{Z}_\ell \subset \mathcal{E}$. After a suitable change of basis, then, $\bar{C}$ can be transformed to a block matrix $(D\,|\,0)$ where D is an $a \times a$ diagonal matrix and 0 denotes the $a \times (b-a)$ zero matrix. The list of diagonal entries of D can be adjusted by a further change of basis to be of the form $(\ell^{k_1}, \dots, \ell^{k_r}, 0, \dots, 0)$, where $r \le a$ is some integer, the k_i's are appropriate exponents, and there are $(a-r)$ terminal zero entries. It is then clear from [33, 3.10] that the Rothenberg-Steenrod spectral sequence of 10.10 collapses with no Hopf algebra extensions and gives a formula for $\mathrm{H}_*\,\Omega_0^\infty X$. Observe in making the calculation that, because F is an infinite loop space and hence homotopy abelian, there is an isomorphism of Hopf algebras

$$\mathrm{H}_*\,F \cong \mathbb{Z}/\ell[\pi_1 X] \otimes_{\mathbb{Z}/\ell} (\otimes^b\,\mathrm{H}_*\,\hat{\mathrm{B}}\mathrm{U})\,.$$

Moreover, the action of the factor $\mathbb{Z}/\ell[\pi_1 X]$ on $\otimes^a\,\mathrm{H}_*\,\hat{\mathrm{B}}\mathrm{U}$ is trivial, and the resulting contribution $\mathrm{Tor}_*^{\mathbb{Z}/\ell[\pi_1 X]}(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$ to the E_2-term is a primitively generated exterior algebra on n generators in bidegree $(0,1)$, where n is the rank of $\pi_1 X$ over $\mathbb{Z}_\ell$.

The formula for $\mathrm{H}_*\,\Omega_0^\infty X$ visibly depends only on $C(0)$. □

Proof of 10.4. Delooping the cofibration sequence of 8.15 gives a cofibration sequence like 10.7 for $\hat{L}(K^{\mathrm{red}}R_0)$. It then follows from a combination of 10.8 and 10.9 that there is an isomorphism of Hopf algebras

$$\mathrm{H}^*\,\Omega_0^\infty \hat{L}(K^{\mathrm{red}}R_0) \cong H(\pi_1\hat{L}(K^{\mathrm{red}}R_0), 1) \otimes H(\pi_0\hat{L}(K^{\mathrm{red}}R_0), 0)\,.$$

The proof is completed by computing the indicated homotopy groups with 6.4 and 3.10. □

The general case. If X and Y are two spectra with an action of Δ_F, define $X \wedge_{\Delta_F} Y$ to be $(X \wedge_\ell Y)^{\Delta_F}$ (see 9.4), where $g \in \Delta_F$ acts on $X \wedge_\ell Y$ by $g \wedge_\ell g^{-1}$. Let i be either 0 or 1, and let N be a finitely generated $\mathbb{Z}_\ell[\Delta_F]$-module which is free as a $\mathbb{Z}_\ell$-module if $i = 1$. Define the Hopf algebra $H_{\Delta_F}(N, i)$ by the formula

$$H_{\Delta_F}(N, i) = \mathrm{H}^* \, \Omega_0^\infty(\mathcal{M}(N, i) \wedge_{\Delta_F} \hat{\mathcal{K}}) \,.$$

The action of Δ_F on $\hat{\mathcal{K}}$ in this formula comes from the usual inclusion $\Delta_F \subset \Delta \subset \Gamma'$.

10.11 Theorem. *There is an isomorphism of Hopf algebras*

$$\mathrm{H}^* \, \Omega_0^\infty \hat{L}(K^{\mathrm{red}} R) \cong H_{\Delta_F}(E_0'(\mathit{red}), 1) \otimes H_{\Delta_F}(A_0', 0) \otimes H_{\Delta_F}(B_0, 0) \,.$$

Remark. By 9.4 the Hopf algebra $H_{\Delta_F}(N, i)$ is a tensor factor of $H(N, i)$. These algebras can be described very explicitly in terms of the algebras A_n and $A_{n,j}$ of [33]. Let $d = |\Delta_F|$. Then $H_{\Delta_F}(\mathbb{Z}_\ell(m), 0)$ is the polynomial Hopf algebra which is the tensor product of the algebras A_n with $(n, \ell) = 1$ and n congruent to $-m$ mod d. The algebra $H_{\Delta_F}(\mathbb{Z}_\ell(m), 1)$ is the exterior algebra which is the “delooping” of $H_{\Delta_F}(\mathbb{Z}_\ell(m), 0)$. Finally, $H_{\Delta_F}(\mathbb{Z}/\ell^k(m), 0)$ can be described in much the same way as $H(\mathbb{Z}/\ell^k, 0)$ is described in 10.5, but with the role of $P(k)$ played by the tensor product of the algebras $A_{n,k-1}$ for $(n, \ell) = 1$ and n congruent to $-m$ mod d. For further details see [33].

10.12 Remark. The method of 10.6 also gives alternate formulations of 10.11. In applying the method it is useful to observe that a short exact sequence of $\mathbb{Z}_\ell[\Delta_F]$ modules splits over $\mathbb{Z}_\ell[\Delta_F]$ if it splits over $\mathbb{Z}_\ell$. For example, combining 9.8, the proof of 9.9, and some elementary algebra shows that there is an isomorphism of Δ_F-modules

$$E_0'(\mathrm{red}) \cong \mathbb{Z}_\ell[\Delta_F]_{\mathrm{ev}}^{r_1(F)} \oplus \mathbb{Z}_\ell[\Delta_F]^{r_2(F)} \oplus B_0 \,.$$

Since $\mathcal{M}(\mathbb{Z}_\ell[\Delta_F]_{\mathrm{ev}}, 1) \wedge_{\Delta_F} \hat{\mathcal{K}}$ is $\Sigma\hat{\mathcal{K}}\mathcal{R}$ (9.11), and $\mathcal{M}(\mathbb{Z}_\ell[\Delta_F], 1) \wedge_{\Delta_F} \hat{\mathcal{K}}$ is $\Sigma\hat{\mathcal{K}}$, the following formula results:

$$\begin{aligned}\mathrm{H}^* \, \Omega_0^\infty \hat{L}(K^{\mathrm{red}} R) \cong{} & \mathrm{H}^*((\mathrm{U}/\mathrm{O})^{r_1(F)} \times \mathrm{U}^{r_2(F)}) \\ & \otimes H_{\Delta_F}(B_0, 1) \otimes H_{\Delta_F}(A_0', 0) \otimes H_{\Delta_F}(B_0, 0) \,.\end{aligned}$$

Here U/O appears because the ℓ-completion of U/O is $\Omega_0^\infty \Sigma\hat{\mathcal{K}}\mathcal{R}$.

We need to set up some notation for the next two lemmas. Suppose that X is a spectrum which lies in a cofibration sequence

$$(\hat{\mathcal{K}})^b \xrightarrow{\ C\ } (\hat{\mathcal{K}})^a \longrightarrow X \qquad (10.13)$$

where C is an $a \times b$ matrix with coefficients in $\Lambda[\Delta_F] \subset \Lambda'$. Write $C(0)$ for the $a \times b$ matrix over $\mathbb{Z}_\ell[\Delta_F]$ obtained by applying to C the augmentation homomorphism $\Lambda \to \mathbb{Z}_\ell$. Let $d = |\Delta_F|$, and let $\tilde{C}$ be the $ad \times bd$ matrix over Λ obtained from C

by identifying $\Lambda[\Delta_F]$ with $(\Lambda)^d$ via the basis consisting of group elements; $\tilde{C}(0)$ is then an $ad \times bd$ matrix over $\mathbb{Z}_\ell$. For notational simplicity write $\mathrm{Map}(\Delta_F, \hat{\mathcal{K}})$ for a product of copies of $\hat{\mathcal{K}}$ indexed by elements of Δ_F, and observe that this has a conjugation action of Δ_F, i.e., $g \cdot f(h) = gf(hg^{-1})$. Define a spectrum $\tilde{X}$ by the cofibration sequence

$$\mathrm{Map}(\Delta_F, \hat{\mathcal{K}})^b \xrightarrow{\tilde{C}} \mathrm{Map}(\Delta_F, \hat{\mathcal{K}})^a \to \tilde{X}\,.$$

so that $\tilde{X}$ inherits an action of Δ_F (see the proof of 9.6).

10.14 Lemma. *Define $\mathbb{Z}_\ell[\Delta_F]$-modules U_0 and U_1 by the exact sequence*

$$0 \to U_1 \to \mathbb{Z}_\ell[\Delta_F]^b \xrightarrow{C(0)} \mathbb{Z}_\ell[\Delta_F]^a \to U_0 \to 0\,.$$

Then U_i is isomorphic as a $\mathbb{Z}[\Delta_F]$-module to $\pi_i\tilde{X}$, $i = 0,\ 1$. If $C = C(0)$, then

$$X \simeq \mathcal{M}(U_1, 1) \wedge_{\Delta_F} \hat{\mathcal{K}} \ \bigvee\ \mathcal{M}(U_0, 0) \wedge_{\Delta_F} \hat{\mathcal{K}}\,.$$

Proof. Both statements are elementary calculations.

10.15 Lemma. *The Hopf algebra $\mathrm{H}^*\,\Omega_0^\infty X$ depends only on $C(0)$.*

Proof. This is very similar to the proof of 10.9. Let $\bar{C}$ denote the map $\otimes^b \mathrm{H}_*\,\mathrm{BU} \to \otimes^a \mathrm{H}_*\,\mathrm{BU}$ obtained when the functor $\Omega_0^\infty(-)$ is applied to C. By [33, 3.9] the map $\bar{C}$ depends only on $C(0)$. Since $\mathbb{Z}_\ell[\Delta_F]$ is a direct product of principal ideal domains, the matrix $C(0)$ and hence the map $\bar{C}$ can be diagonalized. It follows from the spectral sequence argument of [33, 3.10] that the map $\bar{C}$ determines $\mathrm{H}_*\,\Omega_0^\infty X$. □

Proof of 10.11. Let X be the spectrum $\hat{L}(K^{\mathrm{red}}R)$. By 8.3, X lies in a cofibration sequence of the form 10.13 (cf. 8.14, 8.15). It follows from the two lemmas above that there is an isomorphism of Hopf algebras

$$\mathrm{H}_*\,\Omega_0^\infty \hat{L}(K^{\mathrm{red}}R) \cong H_{\Delta_F}(\pi_1\tilde{X}, 1) \otimes H_{\Delta_F}(\pi_0\tilde{X}, 0)\,.$$

The proof is completed by using 8.3 and 6.12 to identify $\tilde{X}$ in this case with $\hat{L}(K^{\mathrm{red}}R_0)$. □

§11. Topological K-theory of $\mathrm{BGL}(R)$

In this section we show how to use 8.3 to compute the ℓ-adic topological K-theory groups $\hat{\mathcal{K}}^*\,\mathrm{BGL}(R)$, or equivalently the groups $\hat{\mathcal{K}}^*(\Omega_0^\infty \hat{K}R)$, at least in some cases. The first step is to reduce the problem to something involving the spectrum $\hat{L}(KR)$.

11.1 Theorem. *The natural map*

$$\Omega_0^\infty \hat{K}R \to \Omega_0^\infty \hat{L}(KR)$$

induces an isomorphism on $\hat{\mathcal{K}}^*$.

This will be proved below. Since (9.7) the spectrum $\hat{L}(KR)$ is a wedge product $\hat{L}(K\mathbb{F}) \vee \hat{L}(K^{\mathrm{red}}R)$ in which the first factor is completely determined (4.18) by $\Gamma'_F \subset \Gamma'$ and the second (9.1) by the Iwasawa module M, Theorem 11.1 in principle gives an approach to computing $\hat{\mathcal{K}}^*\Omega_0^\infty\hat{K}R$. In practice it is not necessarily easy to carry this approach through. There is one case, though, in which Bousfield makes this work unnecessary.

Recall the following conjecture of Iwasawa.

11.2 Conjecture. [23] *The Iwasawa module M is ℓ-torsion free.*

Remark. Iwasawa proved (8.1) that M contains no *finite* Λ_F-submodules. This conjecture goes further and for example rules out $\Lambda_F/\ell \cdot \Lambda_F$ as a submodule. By work of Ferrero and Washington [44, p. 130] the conjecture, which is sometimes expressed in the equivalent form "$\mu = 0$", is known to be true for all abelian extensions of $\mathbb{Q}$.

Now let X be the spectrum $\hat{L}(KR)$. We are interested in two ways of associating to X an object in the category **Pa** of pairs of profinite Λ'-modules. The first construction takes X to $\hat{\mathcal{K}}^\bullet X$, given by

$$\hat{\mathcal{K}}^\bullet X = \langle \hat{\mathcal{K}}^0 X, \hat{\mathcal{K}}^{-1} X\rangle = \langle \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_\ell,\ \Lambda' \otimes_{\Lambda'_F} M\rangle$$

and the second takes X to $\mathrm{H}^\bullet(X, \mathbb{Z}_\ell)$, given by

$$\mathrm{H}^\bullet(X; \mathbb{Z}_\ell(1)) = \left\langle \mathrm{H}^2(P^0X; \mathbb{Z}_\ell(1)),\ \mathrm{H}^1(P^0X; \mathbb{Z}_\ell(1))\right\rangle .$$

The action of Λ' on $\mathrm{H}^i(P^0X; \mathbb{Z}_\ell(1))$ ($i = 1$, 2) is the one induced by the usual action of Λ' on $\mathbb{Z}_\ell(1)$. The edge homomorphism in the Atiyah-Hirzebruch spectral sequence for $\hat{\mathcal{K}}^* X$ gives a homomorphism of pairs

$$\Psi_X : \hat{\mathcal{K}}^\bullet \to \mathrm{H}^\bullet(X; \mathbb{Z}_\ell(1)) .$$

(In checking this it helps to remember that the Λ'-module $\mathbb{Z}_\ell(1)$ is isomorphic to $\pi_2\hat{\mathcal{K}}$ or equivalently to $\hat{\mathcal{K}}^{-1}(\mathrm{pt})$.) In [12, §7] Bousfield has constructed an algebraic functor $\hat{W}_\ell$ from a certain category of maps in **Pa** to a category in which $\hat{\mathcal{K}}^* Z$ lies if Z is an infinite loop space. There is a natural map $\hat{W}_\ell \Psi_X \to \hat{\mathcal{K}}^*\Omega_0^\infty X$. The following theorem is a special case of [12, 8.3].

11.3 Theorem. *Let X be the spectrum $\hat{L}(KR)$. Suppose that Iwasawa's conjecture (11.2) is satisfied for F. Then the map*

$$\hat{W}_\ell \Psi_X \to \hat{\mathcal{K}}^*\Omega_0^\infty X$$

is an isomorphism

Using this theorem requires having an algebraic description of Ψ_X for $X = \hat{L}(KR)$. We will sketch such a description and leave the verifications to the reader. Since X splits as $\hat{L}(K^{\mathrm{red}}R) \vee \hat{L}(K\mathbb{F})$, Ψ_X also splits. Let $N = \hat{\mathcal{K}}^{-1}\hat{L}(K^{\mathrm{red}}R) = \Lambda' \otimes_{\Lambda'_F} M$, and let $N' = \mathrm{Hom}_{\Lambda'}(N, \mathbb{Z}_\ell(1))$. The group $\hat{\mathcal{K}}^0\hat{L}(K^{\mathrm{red}}R)$ is zero, so $\Psi_{\hat{L}(K^{\mathrm{red}}R)}$ amounts to a map

$$N \to \mathrm{H}^1(P^0\hat{L}(K^{\mathrm{red}}R); \mathbb{Z}_\ell(1)) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(\pi_1\hat{L}(K^{\mathrm{red}}R), \mathbb{Z}_\ell(1)) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(N', \mathbb{Z}_\ell(1))$$

where we have computed $\pi_1\hat{L}(K^{\mathrm{red}}R)$ from 4.16. It is clear from a naturality argument what this map is. Now let $N = \hat{\mathcal{K}}^0\hat{L}(K\mathbb{F}) = \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_\ell(0)$ and let $N' = \mathrm{Ext}^1_{\Lambda'}(N, \mathbb{Z}_\ell(1))$. The group $\hat{\mathcal{K}}^{-1}\hat{L}(K\mathbb{F})$ is zero, so $\Psi_{\hat{L}(K\mathbb{F})}$ amounts to a map

$$N \to \mathrm{H}^2(P^0\hat{L}(K\mathbb{F}); \mathbb{Z}_\ell(1)) \cong \mathrm{Ext}^1_{\mathbb{Z}_\ell}(\pi_1\hat{L}(K\mathbb{F}), \mathbb{Z}_\ell(1)) \cong \mathrm{Ext}^1_{\mathbb{Z}_\ell}(N', \mathbb{Z}_\ell(1))$$

where we have used that $\pi_2\hat{L}(K\mathbb{F}) = \mathrm{Hom}_{\Lambda'}(N, \mathbb{Z}_\ell(1))$ is zero. Again by a naturality argument, this map can be described as follows. Pick a finitely generated free resolution $\mathcal{R} = (F_1 \to F_0)$ of N, so that $\mathcal{R}' = \mathrm{Hom}_{\Lambda'}(\mathcal{R}, \mathbb{Z}_\ell(1))$ is a (short) cochain complex with N' as its only cohomology group. Let $\mathcal{R}'' = \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{R}', \mathbb{Z}_\ell)$. By the ordinary universal coefficient theorem there is a natural isomorphism

$$\mathrm{H}_0\,\mathcal{R}'' \cong \mathrm{Ext}^1_{\mathbb{Z}_\ell}(N', \mathbb{Z}_\ell(1))\,.$$

The homomorphisms $F_i \to \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{Hom}_{\Lambda'}(F_i, \mathbb{Z}_\ell(1)), \mathbb{Z}_\ell(1))$ give a homomorphism $\mathcal{R} \to \mathcal{R}''$ of chain complexes, and the induced map $\mathrm{H}_0\,\mathcal{R} \to \mathrm{H}_0\,\mathcal{R}''$ is the one we are looking for.

11.4 Examples. Rather than give a complete description of the functor $\hat{W}_\ell$, we will treat a special case. Suppose that $S = (S^0, S^1)$ is a $\mathbb{Z}/2$-graded, augmented, graded-commutative, profinite algebra over $\mathbb{Z}_\ell$ with augmentation ideal $\tilde{S}$, and that $\theta : S \to S$ is a continuous operation which preserves the grading. Suppose that θ satisfies the additivity relation

$$\theta(a+b) = \theta(a) + \theta(b) - \sum_{i=1}^{\ell-1} \frac{1}{\ell}\binom{\ell}{i} a^i b^{\ell-i} \qquad \text{if } |a| = |b|$$

as well as the multiplicativity relation

$$\theta(ab) = \begin{cases} \theta(a)\theta(b) & \text{if } |a| = |b| = 1 \\ \theta(a)b^\ell + a^\ell\theta(b) + \ell\theta(a)\theta(b) & \text{otherwise} \end{cases}$$

for homogeneous elements a, $b \in S$ of degrees $|a|$ and $|b|$ respectively. Assume in addition that $\theta(1) = 0$ and that $\lim_{n\to\infty} a^n = \lim_{n\to\infty} \theta^n(a) = 0$ for $a \in \tilde{S}$. In this case we will call S a "θ-algebra"; in [12, 7.5] this is called a connective, $\mathbb{Z}/2$-graded, ℓ-adic θ^ℓ-ring. An *action of a profinite group G on a θ-algebra S* is a continuous action of G on S, via ring automorphisms that commute with the operation of θ. Bousfield shows [12] that if X is a connected space with $\mathrm{H}^i(X; \mathbb{Z}_\ell) = 0$ for $i = 1$ and $i = 2$, then the $\mathbb{Z}/2$-graded ring $\hat{\mathcal{K}}^*X$ is in a natural way a θ-algebra with an action of Γ'. This is in effect an alternative way of presenting the conventional λ-algebra structure on $\hat{\mathcal{K}}^*X$. For instance, to obtain ordinary Adams operations from Bousfield's point of view, note (4.4) that the action of Γ' gives ψ^k for $(k, \ell) = 1$, while $\psi^\ell(x)$, for $x \in \hat{\mathcal{K}}^*X$, is determined by the formula $\psi^\ell(x) = x^\ell + \ell\theta(x)$ [12, §2].

Let Φ denote the functor from the category of θ-algebras to the category of profinite $\mathbb{Z}_\ell$-modules given by $\Phi(S) = \tilde{S}$, and let $\hat{T}_\ell$ be the left adjoint of Φ. The functor $\hat{T}_\ell$ can be understood as follows. If A is a $\mathbb{Z}/2$-graded finitely generated free $\mathbb{Z}_\ell$-module with basis $\{x_i\}$, let $S_\ell(A)$ denote the graded symmetric algebra on A over $\mathbb{Z}_\ell$, and $\hat{S}_\ell(A)$ the I-adic completion of $S_\ell(A)$ with respect to the kernel $I \subset S_\ell(A)$ of the map $S_\ell(A) \to \mathbb{Z}_\ell$ sending A to 0. For each $n \geq 1$ let A^n be the n-fold cartesian power of A and $f_n : A^{n+1} \to A^n$ the map which is projection on the first n factors; define $\hat{S}_\ell(A^{\mathbb{N}})$ to be $\lim_n \hat{S}_\ell(A^n)$, where the limit is taken with respect to ring homomorphisms induced by the maps f_n. The ring $\hat{S}_\ell(A^{\mathbb{N}})$ is a completion of the polynomial algebra over $\mathbb{Z}_\ell$ generated by elements $x_{i,n}$, $n \geq 1$.

There is a θ-algebra structure on $\hat{S}_\ell(A^{\mathbb{N}})$ obtained by setting $\theta(x_{i,n}) = x_{i,n+1}$ and using the identities which hold in a θ-algebra to extend θ to the rest of the ring. Bousfield shows that there is an isomorphism of θ-algebras between $\hat{S}_\ell(A^{\mathbb{N}})$ and $\hat{T}_\ell(A)$ (in particular, up to isomorphism the θ-algebra structure on $\hat{S}_\ell(A^{\mathbb{N}})$ does not depend on the choice of basis $\{x_i\}$). If A is a $\mathbb{Z}/2$-graded profinite $\mathbb{Z}_\ell$-module which is an inverse limit $\lim_\alpha A_\alpha$ of finitely generated free $\mathbb{Z}_\ell$-modules, $\hat{T}_\ell(A)$ is isomorphic to $\lim_\alpha \hat{T}_\ell(A_\alpha)$.

If a profinite group G acts continuously on A, then naturality gives an induced action of G on $\hat{T}_\ell(A)$. In particular, if $A = \hat{\mathcal{K}}^*X$ for some spectrum X, then the usual action of Γ' on A makes $\hat{T}_\ell(A)$ into a θ-algebra with an action of Γ'. Bousfield shows that if X is a spectrum such that $\hat{\mathcal{K}}^*X$ is ℓ-torsion free and such that $\mathrm{H}^i(P^0X;\mathbb{Z}_\ell) = 0$ for $i = 1$ and $i = 2$, then there is a natural isomorphism $\hat{\mathcal{K}}^*(\Omega_0^\infty X) \cong \hat{T}_\ell(\hat{\mathcal{K}}^*X)$ of θ-algebras with an action of Γ'. This leads to the following result.

11.5 Proposition. *(Bousfield) Suppose that X is a spectrum such that $\hat{\mathcal{K}}^*X$ is ℓ-torsion free. Assume that π_2X is torsion, and that $\Omega_0^\infty X$ splits as a product $\mathrm{B}\pi_1X \times \Omega_0^\infty(P^1X)$. Then there is a natural isomorphism of $\mathbb{Z}/2$-graded algebras*

$$\hat{\mathcal{K}}^*(\Omega_0^\infty X) \cong \hat{\mathcal{K}}^*(B\pi_1X)\hat{\otimes}T_\ell(\hat{\mathcal{K}}^*X)\,.$$

Remark. It is possible to show (cf. [12]) that the isomorphism of 11.5 respects Adams operations and appropriate λ-operations. However, unlike the isomorphism of 11.3, it does not respect completed Hopf algebra structures.

It is not hard to check that Proposition 11.5 applies if X is one of the spectra $\Sigma\hat{\mathcal{K}}$, $\Sigma\hat{L}(K\mathbb{F})$, $\hat{L}(K\mathbb{F})$, or (if Iwasawa's conjecture 11.2 is satisfied) $\hat{L}(KR)$; in the last case, for instance, the necessary splitting of $\Omega_0^\infty X$ is provided by the natural map

$$\hat{\mathrm{B}}(R^\times) = \hat{\mathrm{B}}\mathrm{GL}_1(R) \to \hat{\mathrm{B}}\mathrm{GL}(R) \simeq \Omega_0^\infty\hat{K}R \to \Omega_0^\infty\hat{L}(KR)\,.$$

What remains is to give a proof of 11.1. Given the remarks in 6.4, this theorem is a consequence of the following proposition.

11.6 Proposition. *Let X be an ℓ-complete spectrum, $Y = \hat{L}X$, and $f : X \to Y$ the natural map. Suppose that f induces isomorphisms $\pi_iX \cong \pi_iY$ for $i = 1$, 2. Then f induces an isomorphism $\hat{\mathcal{K}}^*\Omega_0^\infty Y \cong \hat{\mathcal{K}}^*\Omega_0^\infty X$.*

Proof. Recall from 4.8 that there is an equivalence $\hat{L} \simeq L_{\mathcal{M}_\ell}L_{\mathcal{K}}$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & L_{\mathcal{K}}X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle L_{\mathcal{K}}f} \\ Y & \xrightarrow{\;\simeq\;} & L_{\mathcal{K}}Y \end{array}$$

where the lower arrow is an equivalence because any $\hat{\mathcal{K}}^*$-local spectrum is a fortiori K_*-local. Let F be the fibre of f. Since X and Y are local at ℓ (i.e., q times the identity map is a self-equivalence of X or of Y if $(q,\ell) = 1$), the spectra F and $L_{\mathcal{K}}F$

are local at ℓ. Since $\hat{L}f = L_{\mathcal{M}_\ell} L_{\mathcal{K}} f$ is an equivalence, the spectrum $L_{\mathcal{M}_\ell}(L_{\mathcal{K}}F)$ is contractible, and so $L_{\mathcal{K}}F$ is local away from ℓ in the sense that ℓ times the identity map is a self-equivalence of $L_{\mathcal{K}}F$. It follows that $L_{\mathcal{K}}F$ is a rational spectrum. For any spectrum Z there are isomorphisms $\mathbb{Q}\otimes\pi_j(L_{\mathcal{K}}Z) \cong \mathbb{Q}\otimes\pi_j Z$ [8, 4.11(b)], which in this case give isomorphisms $\pi_j L_{\mathcal{K}}F \cong \mathbb{Q}\otimes\pi_j L_{\mathcal{K}}F \cong \mathbb{Q}\otimes\pi_j F$. In particular, $\pi_1 L_{\mathcal{K}}F = 0$.

The map $\pi_1 L_{\mathcal{K}} f$ is surjective because its composite with $\pi_1 g$ is an isomorphism. It is clear then that the map $\Omega_0^\infty L_{\mathcal{K}} f$ is a map of spaces with a connected rational fibre, and so induces an isomorphism on $\hat{\mathcal{K}}^*$. To complete the proof it is only necessary to show that $\Omega_0^\infty g$ induces an isomorphism on $\hat{\mathcal{K}}^*$. By [9, 3.2] there is a commutative diagram

$$\begin{array}{ccccc} \Omega_0^\infty X & \xrightarrow{e_1} & L_{\mathcal{K}}(\Omega_0^\infty X) & \xrightarrow{e_2} & \Omega_0^\infty(L_{\mathcal{K}}X) \\ & & \downarrow & & \downarrow \\ & & \Omega_0^\infty(\bar{P}_2 X) & \longrightarrow & \Omega_0^\infty(\bar{P}_2 L_{\mathcal{K}} X) \end{array} .$$

in which the composite $e_2 \cdot e_1$ is $\Omega_0^\infty g$, the map e_1 induces an isomorphism on $\hat{\mathcal{K}}^*$, and the square is a homotopy fibre square. Here for a space or spectrum Z, $\bar{P}_2 Z$ is the variant of the second Postnikov section of Z with $\pi_j \bar{P}_2 Z \cong \pi_j Z$ for $j < 2$ and $\pi_2 \bar{P}_2 Z \cong \pi_2 Z/(\operatorname{tors}\pi_2 Z)$. Since $\pi_j L_{\mathcal{K}} f$ is a left inverse for $\pi_j g$, $j = 1, 2$, it follows from the above that $\pi_1 L_{\mathcal{K}} X$ is isomorphic to $\pi_1 X$ and that $\pi_2 L_{\mathcal{K}} X$ is isomorphic to $\pi_2 X \oplus D$, where D, as a quotient of the rational vector space $\pi_2 L_{\mathcal{K}} F$, is a divisible abelian group. Let $\bar{D}$ be the rational vector space obtained by dividing D by its torsion subgroup. Then there are isomorphisms

$$\pi_j \Omega_0^\infty(\bar{P}_2 L_{\mathcal{K}} X) \cong \begin{cases} \pi_1 \Omega_0^\infty(\bar{P}_2 X) & j = 1 \\ \pi_2 \Omega_0^\infty(\bar{P}_2 X) \oplus \bar{D} & j = 2 \\ \pi_j \Omega_0^\infty(\bar{P}_2 X) = 0 & j > 2 \end{cases} .$$

From this it is easy to see that the fibre of the map e_2 is a connected rational space, and so e_2 induces an isomorphism on mod ℓ homology and consequently on $\hat{\mathcal{K}}^*$. □

§12. Examples

We will give five examples to illustrate what the theory of the previous sections looks like in special cases. As in §8, let L_∞ denote $\lim_n \pi_0 \hat{L}(K^{\mathrm{red}} R_n)$. In working out these examples it is useful to keep in mind that there is a short exact sequence of Λ'_F-modules (8.7)

$$0 \to A'_\infty \to L_\infty \to B_\infty \to 0 .$$

Combining 8.10, 9.10, and 7.9 gives another exact sequence of Λ'_F-modules

$$\begin{aligned} 0 \to \operatorname{Ext}^1_{\Lambda_F}(L_\infty, \Lambda_F) &\to M(-1) \\ &\to (\Lambda_F^+)^{r_1(F)} \oplus (\Lambda'_F)^{r_2(F)} \to \operatorname{Ext}^2_{\Lambda_F}(A'_\infty, \Lambda_F) \to 0 \end{aligned} \tag{12.1}$$

in which by 7.7 the group $\operatorname{Ext}^2_{\Lambda_F}(A'_\infty, \Lambda_F)$ vanishes as long as A'_∞ has no finite Λ_F-submodules. The Λ'_F-module structure on these Ext groups is described in 7.6

and 8.11; there is also an explicit calculation in 7.12. Keep in mind the standard idempotents $\epsilon_i \in \mathbb{Z}_\ell[\Delta_F]$, $i = 0, \dots, \ell-2$ from 9.4; these are useful in splitting A'_∞ or M into simpler pieces. We sometimes use a topological generator $\gamma \in \Gamma_F$ to identify Λ_F with $\mathbb{Z}_\ell[[T]]$ (§7), and in this case we let c_0 denote $c_F(\gamma) \in \mathbb{Z}_\ell^\times$.

In the first three examples $\Gamma_F = \Gamma$ and $\Delta_F = \Delta$. In all five cases the Iwasawa module M is ℓ-torsion free, and so 11.1 and 11.3 give explicit formulas for $\hat{\mathcal{K}}^*\Omega_0^\infty(KR) \cong \hat{\mathcal{K}}^* \mathrm{BGL}(R)$ (cf. 11.4).

12.2 Example. Suppose that $F = \mathbb{Q}$ and that ℓ is an odd prime for which Vandiver's conjecture is true [44, p. 78]. There is only one prime above ℓ in F_n $(n \ge -1)$ and this prime is principal, so the groups B_n are trivial and $A'_n = A_n$. Recall that ω is the Teichmuller character (§2). The groups $\epsilon_i A'_0$ vanish for i even (this is the Vandiver conjecture itself) and, for i odd, $\epsilon_i A'_0$ is a cyclic group of order ℓ^{ν_i}, where ℓ^{ν_i} is the exact power of ℓ dividing the generalized Bernoulli number $B_{1,\omega^{-i}}$ [44, 10.15]. From 10.12 we read off

$$\mathrm{H}^*\,\Omega_0^\infty \hat{L}(K^{\mathrm{red}}R) \cong \mathrm{H}^*\,\mathrm{U/O} \otimes (\otimes_i H_{\Delta_F}(\mathbb{Z}/\ell^{\nu_i}(i), 0))$$

where i in the tensor product ranges over odd integers between 0 and $(\ell-2)$. The summand $\epsilon_i A'_\infty$ is zero for i even and isomorphic for i odd to the cyclic Λ-module $\Lambda/(f_i(T))$, where $f_i(T) \in \Lambda$ satisfies $f_i(c_0^s - 1) = L_\ell(s, \omega^{1-i})$, L_ℓ being the ℓ-adic L-function [44, p. 199]. The Iwasawa conjecture is true in this case [44, §7.5], so none of the power series f_i are divisible by ℓ and A'_∞ is $\mathbb{Z}_\ell$-torsion free. From 12.1 there is a direct sum decomposition $M(-1) \cong \Lambda_F^+ \oplus N$ in which for i even $\epsilon_i N$ vanishes and for i odd $\epsilon_i N$ is isomorphic to $\Lambda/(g_i(T))$ with $g_i(T) = f_{\ell-1-i}((1+T)^{-1} - 1)$. It follows that $\hat{L}(KR)$ is equivalent to $\hat{\mathcal{KR}} \vee (\vee_{i\ \mathrm{odd}} C_i)$, where C_i is the cofibre of the self-map of $\epsilon_i \hat{\mathcal{K}}$ represented by $g_i(T)$. We also conclude that $(\mathrm{U/O})\hat{}$ is a retract of $\hat{\mathrm{B}}\mathrm{GL}(R)$.

12.3 Example. Let F be an imaginary quadratic extension of $\mathbb{Q}$, and ℓ an odd prime that satisfies the two conditions

(1) $A_0 = 0$, and
(2) ℓ splits as a product $\pi\bar{\pi}$ in $\mathcal{O}_F$, and π is not an ℓ'th power in the completion of F at $\bar{\pi}$.

One example for $\ell = 5$ is $F = \mathbb{Q}[\sqrt{-1}]$. The Iwasawa theory of such fields is studied by Gold[20]. One finds that $A'_n = 0$ for all n, although $A_n \ne 0$ for $n \ge 1$. Thus $A'_\infty = 0$, and since the primes over ℓ are totally ramified in F_∞/F, $s_\infty = 2$ and $B_\infty = \mathbb{Z}_\ell(0)$. From 12.1 there is a direct sum decomposition $M(-1) \cong \Lambda'_F \oplus \mathbb{Z}_\ell(0)$. Since $\mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_\ell(0), 1) \simeq \Sigma\hat{L}(K\mathbb{F})$ (4.18), this gives a wedge decomposition

$$\hat{L}(KR) \simeq \Sigma\hat{\mathcal{K}} \vee \Sigma\hat{L}(K\mathbb{F}) \vee \hat{L}(K\mathbb{F})\,.$$

We also have that $\hat{\mathrm{U}}$ is a retract of $\hat{\mathrm{B}}\mathrm{GL}(R)$. From 10.12 there is an isomorphism of Hopf algebras

$$\mathrm{H}^*\,\Omega_0^\infty \hat{L}(KR) \cong \mathrm{H}^*(\mathrm{U}) \otimes H_{\Delta_F}(\mathbb{Z}_\ell(0), 0) \otimes H_{\Delta_F}(\mathbb{Z}_\ell(0), 1)\,.$$

12.4 Example. Let $F = \mathbb{Q}[\sqrt{257}]$ and $\ell = 3$. This case has been studied by Greenberg [21]. Here $s_\infty = 1$ and for each n the unique prime above ℓ in F_n is principal, so that $B_\infty = 0$ and $A_n = A'_n$ for all n. The group Δ_F is $\mathbb{Z}/2$, and if N is a $\mathbb{Z}_\ell[\Delta_F]$-module we write $\epsilon_0 N = N^+$ and $\epsilon_1 N = N^-$. It is known that $A_0^+ = \mathbb{Z}/\ell = A_0^-$. Hence from 10.12 we can read off

$$\mathrm{H}^*\,\Omega_0^\infty \hat{L}(K^{\mathrm{red}}R) \cong \mathrm{H}^*(\mathrm{U/O}) \otimes H_{\Delta_F}(A_0, 0) \cong \mathrm{H}^*(\mathrm{U/O}) \otimes H(\mathbb{Z}/\ell, 0)$$

where the isomorphism $H_{\Delta_F}(A_0, 0) = H_{\Delta_F}(\mathbb{Z}/\ell[\Delta_F], 0) \cong H(\mathbb{Z}/\ell, 0)$ is a form of Shapiro's lemma. Greenberg shows that there are isomorphisms

$$A_\infty^- \cong \mathrm{Ext}^1_\Lambda(\Lambda/(T - 27u), \Lambda)(1)$$
$$A_\infty^+ \cong \mathbb{Z}/\ell$$

for some ℓ-adic unit u. It follows from 12.1 that M^+ is isomorphic as a module over $\Lambda = \Lambda_F$ to $\Lambda/(T - 27u)$ and that M^- is a submodule of index 3 in Λ^2. Given this it is easy to show that M^- is isomorphic to $\Lambda \oplus \mathcal{I}$, where $\mathcal{I}$ is the maximal ideal of Λ. In particular, the quotient M^- of M by its Λ-torsion submodule is not a free Λ-module. We have

$$\hat{L}(K^{\mathrm{red}}R_0) \cong \mathcal{M}_{\mathcal{K}}(\Lambda' \otimes_\Lambda M, -1) \cong \hat{\mathcal{K}} \vee W \vee W'$$

where W is the fibre of $(\ell, T) : \Sigma^{-1}(\hat{\mathcal{K}} \vee \hat{\mathcal{K}}) \to \hat{\mathcal{K}}$ and W' is the fibre of the self-map $(T - 27u)$ of $\Sigma^{-1}\hat{\mathcal{K}}$. A similar decomposition for $\hat{L}(K^{\mathrm{red}}R)$ results from taking homotopy fixed points (9.6) of Δ_F.

12.5 Example. Let ℓ be an odd prime and F the maximal real subfield of $\mathbb{Q}(\mu_\ell)$. Here $\Delta_F = \mathbb{Z}/2$. In order to make this example explicit we will assume that ℓ satisfies not only Vandiver's conjecture but also the hypotheses of [44, 10.17] (these hypotheses hold for $\ell < 4 \times 10^6$). As in 12.2, for each n there is one prime above ℓ in F_n and this prime is principal, so that $s_\infty = 1$, $B_\infty = 0$, and $A_n = A'_n$ for $-1 \le n \le \infty$. If k is an ℓ-adic integer let $\nu_\ell(k)$ denote the exponent of the largest power of ℓ that divides k. Washington [44, 10.17] shows that for i even the group $\epsilon_i A_\infty$ is zero, and for i odd $\epsilon_i A_\infty$ is either zero or isomorphic to $\Lambda/(T - \alpha_i)$ where α_i is some ℓ-adic integer with $\nu_\ell(\alpha_i) = 1$. Let $i_1, \ldots, i_q$ denote the set of i such that $\epsilon_i A_\infty \neq 0$. From 10.12 we read off an isomorphism of Hopf algebras

$$\mathrm{H}^*\,\Omega_0^\infty \hat{L}(K^{\mathrm{red}}R) \cong \mathrm{H}^*((\mathrm{U/O})^{(\ell-1)/2}) \otimes H_{\Delta_F}(\mathbb{Z}/\ell(1), 0)^{\otimes q}\,.$$

By 12.1 there is a direct sum decomposition

$$M(-1) \cong (\Lambda_F)^{(\ell-1)/2} \oplus \Lambda/(g_{i_1}(T)) \oplus \cdots \oplus \Lambda/(g_{i_q}(T))$$

with $g_{i_j}(T) = (1+T)^{-1} - 1 - \alpha_{i_j}$. This gives a wedge decomposition

$$\hat{L}(K^{\mathrm{red}}R) \simeq (\hat{\mathcal{K}}\mathcal{R})^{(\ell-1)/2} \vee C_1 \vee \ldots \vee C_q$$

where C_j is the cofibre of the self-map of $\hat{\mathcal{K}}\mathcal{R}$ given by g_{i_j}. The space $(\mathrm{U/O})^{(\ell-1)/2}$ is a retract of $\hat{\mathrm{B}}\mathrm{GL}(R)$.

12.6 Example. Choose the same ℓ as in 12.5, but instead work with the maximal real subfield F of $\mathbb{Q}[\mu_{\ell^{m+1}}]$. Again $\Delta_F = \mathbb{Z}/2$. The modules A_∞ and M are abstractly the same as in 12.5 but are now considered modules over $\Lambda'_F = \mathbb{Z}_\ell[\Delta_F] \otimes \Lambda_F$, where $\Lambda_F = \mathbb{Z}_\ell[[T_m]] \subset \mathbb{Z}_\ell[[T]] = \Lambda$ and $T_m = (1+T)^{\ell^m} - 1$. Choose $i \in \{i_1, \dots, i_q\}$. As a module over Λ_F, the summand $\Lambda/(T-\alpha_i)$ of A_∞ is isomorphic to $\Lambda_F/(T_m - \beta_i)$, where $\beta_i = (1+\alpha_i)^{\ell^m} - 1$ and in particular $\nu_\ell(\beta_i) = m+1$. We obtain an isomorphism of algebras

$$\mathrm{H}^* \hat{L}(K^{\mathrm{red}}R) \cong \mathrm{H}^*((\mathrm{U}/\mathrm{O})^{r_1 F}) \otimes H_{\Delta_F}(\mathbb{Z}/\ell^{m+1}(1), 0)^{\otimes q}$$

with $r_1(F) = \ell^m(\ell-1)/2$, and a wedge decomposition

$$\hat{L}(K^{\mathrm{red}}R) \simeq (\hat{\mathcal{K}}\mathcal{R})^{r_1(F)} \vee C'_1 \vee \dots \vee C'_q$$

where C'_j is the cofibre of the self-map of $\hat{\mathcal{K}}\mathcal{R}$ given by $(1+T_m)^{-1} - 1 - \beta_{i_j}$. The space $(\mathrm{U}/\mathrm{O})^{r_1(F)}$ is a retract of $\hat{\mathrm{B}}\mathrm{GL}(R)$.

§13. Local Fields

In this section we sketch how to extend the theory of the previous sections to ℓ-adic local fields. Let $F = E$ be a finite extension of $\mathbb{Q}_\ell$; up to isomorphism, we can consider F a subfield of $\mathbb{C}$. Most of the notation from §2 carries though to F, e.g., $F_\infty = F(\mu_{\ell^\infty})$, $\mathcal{O}_{F_n}$ is the integral closure of $\mathbb{Z}_\ell$ in F, etc. Note that $R_n = F_n$ (because $\mathcal{O}_{F_n}$ is a local ring) and the class groups A_n and A'_n are trivial. For each n the module B_n is $\mathbb{Z}_\ell$, and B_∞ is $\mathbb{Z}_\ell(0)$ [39, II 5.2]. The Iwasawa module M is the Galois group of the maximal abelian ℓ-extension of F_∞.

There are no residue fields qualified to play the role of $\mathbb{F} = R/\mathcal{P}$ (§2), but there are substitutes for them. By Dirichlet's theorem it is possible to choose a rational prime p which is a topological generator of $c_F(\Gamma'_F) \subset \mathbb{Z}_\ell^\times$. Let $\mathbb{F} = \mathbb{F}_{-1}$ denote the finite field of order p and $\mathbb{F}_n$ $(n \geq 0)$ its extension field of degree $d\ell^n$, where as usual $d = d_F = |\Delta_F|$. Let $\bar{F}$ be the algebraic closure of F and g an element of $\mathrm{Gal}(\bar{F}/F)$ such that the image g' of g in $\Gamma'_F = \mathrm{Gal}(F_\infty/F)$ has $c_F(g') = p$. We will let $g_{-1} = g$ and, for $n \geq 0$, $g_n = g^{d\ell^n}$.

Let $\mathcal{F}_n$ denote the homotopy fibre of the self-map of $\hat{L}(K\bar{F})$ given by $(\mathrm{id} - g_n)$. Since the field F_n is fixed by g_n, there is a natural map $\hat{L}(KF_n) \to \mathcal{F}_n$ of spectra. For $n \geq 0$, denote by S_n the subring $\mathbb{Z}[1/\ell, \mu(F_n)]$ of F_n. Recall that Theorem 5.4 describes certain ring spectrum maps $h_n : \hat{L}(K\mathbb{F}_n) \to \hat{L}(KS_n)$.

13.1 Proposition. *For each $n \geq -1$ the composite map*

$$\hat{L}(K\mathbb{F}_n) \xrightarrow{h_n} \hat{L}(KS_n) \to \hat{L}(KF_n) \to \mathcal{F}_n$$

is a homotopy equivalence.

Proof. By [42], the map $K\bar{F} \to K\mathbb{C}$ induces a homotopy equivalence $\hat{L}(K\bar{F}) \simeq \hat{L}(K\mathbb{C}) \simeq \hat{\mathcal{K}}$. The result then follows relatively easily from 5.6 and 4.18. □

We can now define $K^{\mathrm{red}}F_n$ to be the homotopy fibre of the map $KF_n \to P^{-1}\mathcal{F}_n \simeq \hat{K}\mathbb{F}_n$ (see the proof of 4.18). The results of §5 and §6 go through with the appropriate minor change to 5.4(1), and in particular we obtain 1.8 together with a splitting $\hat{L}(KF) \simeq \hat{L}(K^{\mathrm{red}}F) \vee \hat{L}(K\mathbb{F})$.

It is interesting to use the techniques of §8 and §9 to compute the Iwasawa module M. Let r be the degree of F over $\mathbb{Q}_\ell$.

13.2 Proposition. *There is an isomorphism* $M \cong \mathbb{Z}_\ell(1) \oplus (\Lambda'_F)^r$.

Proof. This is in three steps. Let $E_0(\mathrm{red})$ be the kernel of the map

$$\mathcal{O}_{F_0}^\times = \pi_1 K\mathcal{O}_{F_0} \to \pi_1 KF_0 \to \pi_1 \mathcal{F}_0 \,.$$

The first step is to show that for $0 \le i \le (d_F - 1)$ there are equalities

$$\mathrm{rank}_{\mathbb{Z}/\ell}\, \epsilon_i(\mathbb{Z}/\ell \otimes E_0(\mathrm{red})) = r \,.$$

By choice of $\mathcal{F}_0$, $\mathbb{Z}/\ell \otimes E_0(\mathrm{red})$ is isomorphic as a Δ_F-module to $\mathbb{Z}/\ell \otimes U$, where U is the quotient of $\mathcal{O}_{F_0}^\times$ by its torsion subgroup. However, the ℓ-adic logarithm [44, §5.1] shows that U is isomorphic as a Δ_F-module to the additive group $\mathcal{O}_{F_0}$, so that there are equalities

$$\mathrm{rank}_{\mathbb{Z}/\ell}\, \epsilon_i(\mathbb{Z}/\ell \otimes U) = \mathrm{rank}_{\mathbb{Z}_\ell}\, \epsilon_i U = \mathrm{rank}_{\mathbb{Q}_\ell}\, \epsilon_i(\mathbb{Q} \otimes \mathcal{O}_{F_0}) = r$$

for each i between 0 and $(d_F - 1)$, where the last equality comes from the normal basis theorem. The second step is to show that for $0 \le i \le (d_F - 1)$ there are equalities

$$\mathrm{rank}_{\Lambda_F}\, M(-1) = r \,.$$

This follows immediately from the arguments in the proof of 9.9, since there are isomorphisms (6.4, 3.10) of Δ_F-modules

$$\begin{aligned} \pi_1 \hat{L}(K^{\mathrm{red}} F_0) &\cong E'_0(\mathrm{red}) \cong \mathbb{Z}_\ell(0) \oplus E_0(\mathrm{red}) \\ \pi_0 \hat{L}(K^{\mathrm{red}} F_0) &\cong \mathrm{H}^2_{\text{ét}}(F_0; \mathbb{Z}_\ell(1)) \cong B_0 \cong \mathbb{Z}_\ell(0) \end{aligned} \,.$$

Finally, using the argument of 8.17 we calculate that $\mathrm{Hom}_{\Lambda_F}(E'_\infty(\mathrm{red}), \Lambda_F)$ is isomorphic over Λ'_F to $(\Lambda'_F)^d$, and obtain the desired formula from 8.10. $\square$

This immediately gives

13.3 Theorem. *There are wedge decompositions*

$$\begin{aligned} \hat{L}(K^{\mathrm{red}} F) &\simeq (\Sigma \hat{\mathcal{K}})^r \vee \Sigma \hat{L}(K\mathbb{F}) \\ \hat{L}(KF) &\simeq (\Sigma \hat{\mathcal{K}})^r \vee \Sigma \hat{L}(K\mathbb{F}) \vee \hat{L}(K\mathbb{F}) \end{aligned} \,.$$

The method of §9 extends immediately to give an unstable splitting theorem.

13.4 Theorem. *The space* $(\hat{U})^r$ *is a retract of* $\hat{\mathrm{B}}\mathrm{GL}(F)$.

The homology of $\Omega_0^\infty \hat{L}(K^{\mathrm{red}} F)$ is easy to read off from §10. Since M is ℓ-torsion free, 11.1 and 11.3 give a formula for $\hat{\mathcal{K}}^* \Omega_0^\infty(KF) \cong \hat{\mathcal{K}}^* \mathrm{BGL}(F)$ (see 11.4).

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