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STRONG CONVERGENCE OF THE EILENBERG–MOORE SPECTRAL SEQUENCE

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LET $p : E \to B$ be a fibration of pointed spaces with fiber F. Let A be any abelian group, and suppose that the base B is connected. Our main result is:

THEOREM. *The* mod A *Eilenberg–Moore spectral sequence of* p *converges strongly to* $H_*(F, A)$ *if and only if* $\pi_1(B)$ *acts nilpotently on* $H_i(F, A)$ *for each* $i \geq 0$.

This statement has to be explained. First of all, the theorem refers to the general "Eilenberg–Moore" spectral sequence of p, with arbitrary coefficients, constructed in §1. Secondly, *strong convergence* of this second-quadrant spectral sequence means that

(1) for each pair (i, j) such that $j + i \geq 0$, $i \leq 0$, there is an R with the property that

$$E^R_{i,j} = E^\infty_{i,j};$$

(2) for all $n \geq 0$, $\{E^\infty_{i,j} : i + j = n\}$ is the set of filtration quotients from a *finite* filtration of $H_n(F, A)$.

Lastly, the action of a group π on an abelian group M is said to be *nilpotent* (i.e. M is a nilpotent π-module) if there is a *finite* π-filtration of M with the property that π acts trivially on the filtration quotients. In other words, a nilpotent π-module is one which can be constructed from trivial π-modules by a finite number of extensions.

The motivation for the proof below comes from an old idea, due apparently to Adams, for proving the convergence of the rational cobar spectral sequence. The idea consisted in filtering an auxiliary cobar construction to get the Serre spectral sequence, and applying the Zeeman comparison theorem. Here a geometric varient of the cobar construction is used (see §1) and the spectral sequence comparison techniques of Bousfield and Quillen replace the classical Zeeman result. In addition, "pro" arguments are used to avoid the extraneous $\varprojlim$ problems that can arise when B is not simply connected.

Previous work on the convergence of the Eilenberg–Moore spectral sequence has been done by Eilenberg–Moore [5], Larry Smith [13], Alex Heller [7], and V. K. A. M. Gugenheim [6], among others. Some of these authors have obtained results in the case in which $\pi_1(B)$ acts trivially on $H_*(F, A)$.

This paper is written *simplicially*, so that space = simplicial set. Anything which is taken for granted can be found in [8], [2], [9], or [14].

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§1. CONSTRUCTION OF THE EILENBERG–MOORE SPECTRAL SEQUENCE

Given a fiber map $p : E \to B$ of pointed spaces and an abelian group A, this paragraph constructs a mod A Eilenberg–Moore homology spectral sequence, as the *homotopy* spectral sequence of an explicit tower of fibrations. Bousfield–Kan's "cosimplicial" machinery is needed to carry out the construction in purely geometric terms.

We first recall the concept of cosimplicial space, then show how a cosimplicial space gives rise to a tower of ordinary spaces, and finally examine the particular tower that yields the Eilenberg–Moore spectral sequence.

1.1. A *cosimplicial space* $\mathbf{C}$ is a family of spaces $\{\mathbf{C}^n : n \geq 0\}$ together with "coface" and "codegeneracy" maps:

$$d^i : \mathbf{C}^n \to \mathbf{C}^{n+1} \qquad 0 \leq i \leq n+1$$
$$s^i : \mathbf{C}^n \to \mathbf{C}^{n-1} \qquad 0 \leq i \leq n-1$$

which satisfy the dual of the usual simplicial identities. An *augmentation* of a cosimplicial space $\mathbf{C}$ is a map

$$\eta : X \to \mathbf{C}^0$$

such that $d^0\eta = d^1\eta : X \to \mathbf{C}^1$.

1.2. An example is given by the *cosimplicial standard simplex* $\mathbf{\Delta}$, which is the cosimplicial space with

$$\mathbf{\Delta}^n = \Delta[n] \qquad n \geq 0$$

(the standard n-simplex) and with coface and codegeneracy operators induced by the standard inclusions $\Delta[n] \to \Delta[n+1]$ and the standard collapsings $\Delta[n] \to \Delta[n-1]$.

1.3. As another example, the *geometric cobar construction* [12] on a fibration $p : E \to B$ of pointed spaces is the cosimplicial space $\mathbf{F}$ with

$$\mathbf{F}^n = B^n \times E \qquad n \geq 0$$

where

$$d^0(b_1, \ldots, b_n, e) = (*, b_1, \ldots, b_n, e)$$
$$d^{n+1}(b_1, \ldots, b_n, e) = (b_1, \ldots, b_n, p(e), e)$$
$$d^i(b_1, \ldots, b_n, e) = (b_1, \ldots, b_i, b_i, \ldots, b_n, e) \qquad 1 \leq i \leq n$$
$$s^i(b_1, \ldots, b_n, e) = (b_1, \ldots, b_i, b_{i+2}, \ldots, e) \qquad 0 \leq i \leq n-1.$$

Here $*$ is the basepoint of B. Note that if F is the fibre of p, $\mathbf{F}$ is augmented by the natural inclusion

$$F \to E = \mathbf{F}^0.$$

In some sense a cosimplicial space is just a rigidly structured inverse system of spaces. An explicit connection between cosimplicial spaces and inverse systems is provided by Bousfield–Kan's family of "Tot" functors.

1.4. For each $s \geq 0$, Tot_s is the functor from *cosimplicial spaces* to *spaces* given by

$$(\mathrm{Tot}_s\, \mathbf{C})_k = \mathrm{Hom}(\mathbf{\Delta}_{(s)} \times \Delta[k], C).$$

Here $(\mathrm{Tot}_s\, \mathbf{C})_k$ is the set of k-simplicies of the space $\mathrm{Tot}_s\, \mathbf{C}$, $\mathbf{\Delta}_{(s)}$ is the s-skeleton of the cosimplicial standard simplex, and "Hom" indicates Hom in the category of cosimplicial spaces. The simplicial structure of $\mathrm{Tot}_s\, \mathbf{C}$ comes in the usual way from the standard maps $\Delta[k] \to \Delta[k+1]$ and $\Delta[k] \to \Delta[k = 1]$.

Note that the inclusions $\mathbf{\Delta}_{(s)} \to \mathbf{\Delta}_{(s+1)}$ induce maps $\mathrm{Tot}_{s+1}\, \mathbf{C} \to \mathrm{Tot}_s\, \mathbf{C}$ which form the family of spaces $\{\mathrm{Tot}_s\, \mathbf{C} : s \geq 0\}$ into a *tower*:

$$\cdots \to \mathrm{Tot}_{s+1}\, \mathbf{C} \to \mathrm{Tot}_s\, \mathbf{C} \to \cdots \to \mathrm{Tot}_0\, \mathbf{C} = \mathbf{C}^0.$$

Furthermore, if $\mathbf{C}$ is augmented by $X \to \mathbf{C}^0$, then the tower $\{\mathrm{Tot}_s\, \mathbf{C}\}$ is also augmented in the sense that there are maps

$$X \to \mathrm{Tot}_s\, \mathbf{C} \qquad s \geq 0$$

which give rise to commutative diagrams:

$$\begin{array}{ccc} & \nearrow & \mathrm{Tot}_{s+1}\mathbf{C} \\ X & & \downarrow \\ & \searrow & \mathrm{Tot}_s\mathbf{C}. \end{array}$$

The inverse limit of the tower $\{\mathrm{Tot}_s\, \mathbf{C}\}$, denoted $\mathrm{Tot}\, \mathbf{C}$, plays a large role in the theory of completions and in the theory of homotopy inverse limits. However, we are interested in towers $\{\mathrm{Tot}_s\, \mathbf{C}\}$ themselves, rather than in their inverse limits.

1.5. One more preliminary definition is needed. Let A be an abelian group, and X a space; then $A \otimes X$ is the *space* given by

$$(A \otimes X)_k = \bigoplus_{x \in X_k} A. \qquad k \geq 0.$$

Here $\oplus$ denotes direct sum of abelian groups. The face and degeneracy operators of $A \otimes X$ are induced in the standard way from those of X. It is clear how to prolong $A \otimes$ to a functor on the category of cosimplicial spaces by the technique of applying it in each codimension.

The most useful property of $A \otimes$ on the category of spaces is that there is a natural isomorphism

$$\pi_*(A \otimes X) = H_*(X, A). \tag{1.6}$$

Thus, with the help of $A \otimes$, homological questions can be reduced to homotopical ones.

All the background has now been filled in for the definition of the Eilenberg–Moore spectral sequence with coefficients in A. Let $p : E \to B$ be a fibration of pointed spaces, with fiber F, and let $\mathbf{F}$ be the cosimplicial space of (1.3).

1.7. The *mod A Eilenberg–Moore spectral sequence of p* is the homotopy spectral sequence of the tower of fibrations

$$\cdots \to \mathrm{Tot}_s(A \otimes \mathbf{F}) \to \mathrm{Tot}_{s-1}(A \otimes \mathbf{F}) \to \cdots \to \mathrm{Tot}_0(A \otimes \mathbf{F})$$

(see [1] for a similar definition). The fact that this is indeed a tower of fibrations follows from [2]. In view of (1.3) and (1.5), the tower is augmented by

$$A \otimes F \to \{\mathrm{Tot}_s(A \otimes \mathbf{F})\}.$$

The E^2 term of this spectral sequence can be computed according to [2] under suitable flatness assumptions, for instance, when A is the additive group of a field. It can be shown using [3] and [12] that the spectral sequence coincides from E^2 onward with the spectral sequences of [13], [12] and [5], whenever these other spectral sequences are defined.

§2. FUNDAMENTAL GROUP ACTIONS

Proof of the "Only If" Part

In §1 the mod A Eilenberg–Moore spectral sequence of $p : E \to B$ was constructed as the homotopy spectral sequence of a tower of fibrations. This paragraph defines an *action* of $\pi_1(B)$ upon the homotopy groups of the spaces in that tower. It turns out that

(2.1) this action is *nilpotent* at each stage, that is, $\pi_1(B)$ acts *nilpotently* on $\pi_i\mathrm{Tot}_s(A \otimes \mathbf{F})$ for $i, s \geq 0$;

(2.2) the action is compatible, via the augmentation $A \otimes F \to \{\mathrm{Tot}_s(A \otimes \mathbf{F})\}$, with the usual action of $\pi_1(B)$ on $\pi_i(A \otimes F) = H_i(F, A)$.

This enables us to prove half of the main theorem:

2.3. *Let $p : E \to B$ be a fibration of pointed spaces, with fiber F. Suppose that B is connected. If the* mod A *Eilenberg–Moore spectral sequence of p converges strongly to $H_*(F, A)$, then each $H_i(F, A)$, $i \geq 0$, is a nilpotent $\pi_1(B)$ module.*

The proof is straightforward. Strong convergence implies, among other things, that for each i there is an s such that the augmentation $H_i(F, A) \to \pi_i\,\mathrm{Tot}_s(A \otimes \mathbf{F})$ is injective. Thus $H_i(F, A)$, as a submodule of a nilpotent $\pi_1(B)$ module, is itself nilpotent.

It remains to define the $\pi_1(B)$ action on $\pi_i\,\mathrm{Tot}_s\,(A \otimes \mathbf{F})$ and to prove (2.1) and (2.2). Before doing this, it is necessary to generalize slightly the "geometric cobar construction" of (1.3) in a way which will be useful at several junctures later on.

2.4. The "*two-sided geometric cobar construction*" on a fixed map $p : E \to B$ and any other map $f : X \to B$ is the cosimplicial space $\mathbf{C}(f)$ defined by the formula

$$\mathbf{C}(f)^n = X \times B^n \times E \qquad n \geq 0$$

where

$$\begin{aligned} d^0(x, b_1, \ldots, b_n, e) &= (x, f(x), b_1, \ldots, b_n, e) \\ d^{n+1}(x, b_1, \ldots, b_n, e) &= (x, b_1, \ldots, b_n, p(e), e) \\ d^i(x, b_1, \ldots, b_n, e) &= (x, b_1, \ldots, b_i, b_i, \ldots, b_n, e) \qquad 1 \leq i \leq n \\ s^i(x, b_1, \ldots, b_n, e) &= (x, b_1, \ldots, b_i, b_{i+2}, \ldots, b_n, e) \qquad 0 \leq i \leq n-1. \end{aligned}$$

$\mathbf{C}(f)$ is augmented by the natural inclusion of $X \times_B Y \to X \times Y = \mathbf{C}(f)^0$. Furthermore, using the obvious notation,

$$\mathbf{F} = \mathbf{C}(* \to B).$$

It is now possible to give the action of $\{\alpha\}$ in $\pi_1(B)$ on $\pi_i \operatorname{Tot}_s(A \otimes \mathbf{F})$. Assume without loss of generality that B satisfies the extension condition, so that $\alpha : \Delta[1] \to B$ can be chosen to represent $\{\alpha\}$. The two face inclusions

$$\delta^0, \delta^1 : \Delta[0] \to \Delta[1]$$

evidently induce two maps of cosimplicial spaces

$$\mathbf{F} \xrightarrow{\delta^0} \mathbf{C}(\alpha) \xleftarrow{\delta^1} \mathbf{F}.$$

Since $\delta^i : \mathbf{F} \to \mathbf{C}(\alpha)$ $(i = 0, 1)$ is a weak homotopy equivalence in each codimension, it follows from [2] that the induced map

$$\operatorname{Tot}_s(A \otimes \delta_i) : \operatorname{Tot}_s(A \otimes \mathbf{F}) \to \operatorname{Tot}_s(A \otimes \mathbf{C}(\alpha)) \qquad (i = 0, 1)$$

is a weak homotopy equivalence for each $s \geq 0$. The action of α on $\pi_i(\operatorname{Tot}_s(A \otimes \mathbf{F}))$ is defined to be via the automorphism:

$$(\pi_i(\operatorname{Tot}_s(A \otimes \delta^1))^{-1}(\pi_i \operatorname{Tot}_s(A \otimes \delta^0)).$$

It is easy to see that this automorphism is well-defined (because $\operatorname{Tot}_s(A \otimes \mathbf{F}))$ is a simple space) and depends only on the homotopy class $\{\alpha\}$ of α. If F is the fiber of $p : E \to B$, there is a straightforward way to obtain the usual action of $\pi_1(B)$ on $\pi_i(A \otimes F)$ with a construction almost identical to the one above. This proves the naturality statement (2.2). The fact that $\pi_i \operatorname{Tot}_s(A \otimes \mathbf{F})$ is a nilpotent $\pi_1(B)$ module for all i, $s \geq 0$ follows easily by induction on s, using the standard formula [2] for the homotopy of the fiber of $\operatorname{Tot}_s(A \otimes \mathbf{F}) \to \operatorname{Tot}_{s-1}(A \otimes \mathbf{F})$, together with the fact that $\pi_1(B)$ acts trivially on the homology of the fibre in the product fibration over B with fiber $B^n \times E$.

§3. TOWER ALGEBRA

This paragraph recalls the basic algebraic properties of towers of abelian groups, and interprets strong convergence of the Eilenberg–Moore spectral sequence in the language of these towers. This re-formulation of the main theorem is necessary for the proof in §4.

A *tower* $\{G_s\}$ of abelian groups is as usual a non-negatively indexed family of abelian groups $\{G_s : s \geq 0\}$ together with maps $G_{s+1} \to G_s$. A morphism $f : \{G_s\} \to \{H_s\}$ of towers is an assemblage of maps $f_s : G_s \to H_s$ such that all the diagrams

$$\begin{array}{ccc} G_{s+1} & \to & H_{s+1} \\ \downarrow & & \downarrow \\ G_s & \to & H_s \end{array}$$

commute. Basic to tower theory is the notion of tower morphisms which are isomorphisms "in the limit", although not necessarily at each level. This is made precise in the following definition:

3.1. A morphism $f: \{G_s\} \to \{H_s\}$ is called a *pro-isomorphism* iff for every abelian group K the induced map

$$\varinjlim_s \operatorname{Hom}(H_s, K) \to \varinjlim_s \operatorname{Hom}(G_s, K)$$

is an isomorphism.

This elegant definition can be put into a more concrete form. Call a tower $\{G_s\}$ *pro-trivial* if for every s there is an $s' \geq s$ such that the map $G_{s'} \to G_s$ is trivial. It is a simple algebraic exercise to show

3.2 [2] *A morphism* $f: \{G_s\} \to \{H_s\}$ *is a pro-isomorphism iff both of the towers* $\{\ker f_s\}$ *and* $\{\operatorname{coker} f_s\}$ *are pro-trivial.*

A careful look at the definitions now justifies the following reformulation of "strong convergence".

3.3. *Let* $p: E \to B$ *be a fibration of pointed spaces with fiber* F. *The* mod A *Eilenberg–Moore spectral sequence of p converges strongly to* $H_*(F, A) = \pi_*(A \otimes F)$ *in the sense of the introduction iff the augmentation*

$$A \otimes F \to \{\operatorname{Tot}_s(A \otimes \mathbf{F})\}$$

induces pro-isomorphisms

$$\pi_i(A \otimes F) \to \{\pi_i \operatorname{Tot}_s(A \otimes \mathbf{F})\} \qquad i \geq 0.$$

(The group $\pi_i(A \otimes F)$ is meant to be considered as a constant tower.)

§4. COMPLETION OF THE PROOF

The "If" Part

Suppose that $p: E \to B$ is as usual a fibration of pointed spaces with fiber F. In this paragraph we show that if B is connected and $\pi_1(B)$ acts nilpotently on each $H_i(F, A)$, the mod A Eilenberg–Moore spectral sequence of p converges strongly to $H_*(F, A)$.

The technique is to construct a tower of *first-quadrant* spectral sequences, augmented by the Serre spectral sequence of p. Recall that this Serre spectral sequence ties together the homology of B, the homology of E, and the homology of F. The new tower will tie these first two homologies to the unknown abutment of the Eilenberg–Moore spectral sequence. Under nilpotency assumptions, a comparison lemma will imply that the unknown abutment is in fact isomorphic to the homology of F.

First of all, it is necessary to introduce an auxiliary cosimplicial space $\mathbf{E}$, which will, roughly speaking, play the role of the "total space of a fibration over B with fiber $\mathbf{F}$". In the notation of §2, $\mathbf{E}$ is given as $C(i)$, where $i: B \to B$ is the identity map. It is important to notice $\mathbf{E}$, unlike $\mathbf{F}$, is always homologically well-behaved. In fact,

4.1. *There is a natural augmentation map* $E \to \mathbf{E}$. *The induced map*

$$A \otimes E \to \{\operatorname{Tot}_s(A \otimes \mathbf{E})\}$$

gives rise, for each $i \geq 0$, *to a pro-isomorphism* $\pi_i(A \otimes E) \to \{\pi_i \operatorname{Tot}_s(A \otimes \mathbf{E})\}$.

Proof. The existence of the augmentation map follows from §2. The rest follows from [2], since the obvious inverse to $i : B \to B$ induces a *contraction* of the cosimplicial space $\mathbf{E} = \mathbf{C}(\imath)$ upon its augmentation.

Now, for each $k \geq 0$, let $i_{(k)} : B_{(k)} \to B$ be the inclusion of the k-skeleton. Let $\mathbf{E}(k) = \mathbf{C}(\imath_{(k)})$ and let $E(k) = B_{(k)} \times_B E$, that is, $E(k) = p^{-1}(B_{(k)})$. It is well known that the Serre spectral sequence E^r of p arises from the filtration of $A \otimes E$ given by

$$A \otimes E(0) \to A \otimes E(1) \to \cdots \to A \otimes E.$$

Along the same lines, for each $s \geq 0$ there is a spectral sequence E_s^r for $\pi_* \operatorname{Tot}_s(A \otimes \mathbf{E})$ arising from the filtration of $\operatorname{Tot}_s(A \otimes \mathbf{E})$ given by

$$\operatorname{Tot}_s(A \otimes \mathbf{E}(0)) \to \operatorname{Tot}_s(A \otimes \mathbf{E}(1)) \to \cdots \to \operatorname{Tot}_s(A \otimes \mathbf{E}).$$

It is clear that these spectral sequences E_s^r form a tower $\{E_s^r : s \geq 0\}$. The commutative diagram

$$\begin{array}{ccccccccc} E(0) & \to & E(1) & \to & E(2) & \to & \cdots & \to & E \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \mathbf{E}(0) & \to & \mathbf{E}(1) & \to & \mathbf{E}(2) & \to & \cdots & \to & \mathbf{E} \end{array}$$

where the vertical maps are augmentations, shows that this tower is augmented by the Serre spectral sequence of p.

It is not immediately clear that E_s^r is a first quadrant spectral sequence; however, E_s^r does converge, since the given filtration of $\operatorname{Tot}_s(A \otimes \mathbf{E})$ is finite in each dimension. This follows from the various definitions together with the observation that $\Delta_{(s)} \times \Delta[k]$ is generated, as a cosimplicial-simplicial set, by its elements of codimension $\leq s$ and dimension $\leq s + k$.

Most of the usefulness of the spectral sequence E_s^r comes from the fact that it is possible to compute its E^2 term:

4.2a. *E_s^r has the "Serre" E^2 term, that is,*

$$E_s^2(i, j) = H_i(B, \pi_j \operatorname{Tot}_s(A \otimes \mathbf{F})).$$

4.2b. *The augmentation map $E^2 \to E_s^2$ is the obvious one induced by the $\pi_1(B)$ equivarient augmentation $\pi_*(A \otimes F) \to \pi_* \operatorname{Tot}_s(A \otimes \mathbf{F})$.*
This computation is carried out in §5.

The following two algebraic results (due to Bousfield) allow the Serre spectral sequence E^r of p to be compared to the tower $\{E_s^r\}$ "in the limit". The results are mainly consequences of the fact that pro-isomorphisms satisfy a strong analogue of the "five lemma".

4.3. Tower Comparison Lemma [2: III, 7.2]. *Let*

$$\{E_s^r(i, j)(X) \Rightarrow H_{i+j}(X_s)\} \xrightarrow{f} \{E_s^r(i, j)(Y) \Rightarrow H_{i+j}(Y_s)\}$$

be a map of towers of first quadrant spectral sequences of homological type. If $H_n(f)$ is a pro-isomorphism for all n, and $E^2(i, j)(f)$ is a pro-isomorphism for $j < k$, then

(a) *$E^2(0, k)(f)$ is a pro-isomorphism;*

(b) *$E^2(1, k)(f)$ is a pro-epimorphism (that is, its cokernel is pro-trivial).*

Just as the above is a straightforward tower version of the Zeeman comparison theorem, Bousfield's second result is the tower version of a well-known lemma about nilpotent actions (see [4], [10], [11]).

4.4. A Consequence of Nilpotency. *Let π be a group, and $f: \{M_s\} \to \{N_s\}$ a map between two towers of nilpotent π-modules. If $H_0(\pi, f)$ is a pro-isomorphism, and $H_1(\pi, f)$ is a pro-epimorphism, then f itself is a pro-isomorphism.*

Now all is in readiness to complete the proof of the main theorem.

Assume that $\pi_1(B)$ acts nilpotently on each $H_i(F, A)$. We will show, by induction on n, that for each n the augmentation map

$$\pi_n(A \otimes F) \to \{\pi_n \operatorname{Tot}_s(A \otimes \mathbf{F})\}$$

is a pro-isomorphism. By §3, this is equivalent to the promised strong convergence result.

Consider the augmentation of the tower

$$\{E_s^r(i, j) \Rightarrow \pi_{i+j} \operatorname{Tot}_s(A \otimes \mathbf{E})\}$$

by the constant tower consisting of the Serre spectral sequence of p:

$$E^r(i, j) \Rightarrow \pi_{i+j}(A \otimes E).$$

By (4.1), the induced "E^∞" maps

$$\pi_i(A \otimes E) \to \{\pi_i \operatorname{Tot}_s(A \otimes \mathbf{E})\}$$

are pro-isomorphisms for each $i \geq 0$. By induction, choose n such that the maps

$$\pi_i(A \otimes F) \to \{\pi_i \operatorname{Tot}_s(A \otimes \mathbf{F})\}$$

are pro-isomorphisms for $i < n$. (This is certainly true, by default, for $n = 0$.) The comparison lemma (4.3) then says that

(a) $H_0(B, \pi_n(A \otimes F)) \to \{H_0(B, \pi_n \operatorname{Tot}_s(A \otimes \mathbf{F}))\}$ is a pro-isomorphism;

(b) $H_1(B, \pi_n(A \otimes F) \to \{H_1(B, \pi_n \operatorname{Tot}_s(A \otimes \mathbf{F}))\}$ is a pro-epimorphism.

Write π for $\pi_1(B)$. It follows at once from (a) and (b) that

(a)′ $H_0(\pi, \pi_n(A \otimes F)) \to \{H_0(\pi, \pi_n \operatorname{Tot}_s(A \otimes \mathbf{F}))\}$ is a pro-isomorphism;

(b)′ $H_1(\pi, \pi_n(A \otimes F)) \to \{H_1(\pi, \pi_n \operatorname{Tot}_s(A \otimes \mathbf{F}))\}$ is a pro-epimorphism.

Since π acts nilpotently both on $\pi_n(A \otimes F)$ and on each $\pi_n \operatorname{Tot}_s(A \otimes \mathbf{F})$, $s \geq 0$ (the first by hypothesis, the second by §2), it follows from (4.4) that

$$\pi_n(A \otimes F) \to \{\pi_n \operatorname{Tot}_s(A \otimes \mathbf{F})\}$$

is a pro-isomorphism. This carries the induction one step further, and completes the proof of the main theorem.

§5. COMPUTATIONAL APPENDIX

This paragraph is concerned with computing the E^2 terms of the family of first quadrant spectral sequences introduced in §4. The process is almost the same as that of computing the E^2 term of the ordinary Serre spectral sequence.

Recall that it is necessary to show that

$$E_s^2(i,j) = H_i(B, \pi_j \operatorname{Tot}_s(A \otimes \mathbf{F}))$$

and also that this identification is naturally related to the standard identification of the E^2 term of the Serre spectral sequence. We will concentrate on computing $E_s^1(i,j)$: the differential d^1 can then be evaluated in the usual way. The naturality statement will follow spontaneously from the form of the argument.

A relativization of the notation of §2 will come in handy below. Namely, if

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ {\scriptstyle j}\uparrow & \nearrow_{g} & \\ Y & & \end{array}$$

is a commutative diagram in which j is a cofibration (that is, an injection of simplicial sets), then $\mathbf{C}(f, g)$ is defined to be the cosimplicial (pointed) space which in each codimension k is the cofibre of the map

$$\mathbf{C}(g)^k \to \mathbf{C}(f)^k$$

induced by j. Moreover, if $\mathbf{C}$ is any cosimplicial pointed space, the "reduced chain space of $\mathbf{C}$", $A \underline{\otimes} \mathbf{C}$, is by definition the cosimplicial space given in each codimension k by the obvious quotient of $A \otimes \mathbf{C}^k$, i.e.

$$(A \underline{\otimes} \mathbf{C})^k = (A \otimes (\mathbf{C}^k))/(A \otimes *).$$

As in the Serre case, the main idea in the computation of E_s^1 is to use a sort of "local triviality" to justify invoking a Kunneth-like formula over each non-degenerate simplex of B.

Let $b \in B_n$ be a non-degenerate n-simplex. Its characteristic map $\Delta[n] \to B$ will be denoted by the same letter (so $b(\Delta_n) = b$), and ∂b will denote the restriction of b to $\dot{\Delta}[n]$. The standard n-sphere, S^n, is $\Delta[n]/\dot{\Delta}[n]$, and $\Sigma^n : S^n \to B$ is the unique map which sends everything to the basepoint.

5.1 (Local Triviality). *There is a natural isomorphism:*

$$\pi_* \operatorname{Tot}_s(A \underline{\otimes} \mathbf{C}(b, \partial b)) = \pi_* \operatorname{Tot}_s(A \underline{\otimes} \mathbf{C}(\Sigma^n, *)).$$

Proof. Assume without loss of generality that B has only one zero-simplex. Let $H^b : \Delta[n] \times \Delta[1] \to B$ be the null homotopy of the map b induced by the contraction of $\Delta[n]$ upon its first vertex. $H_0{}^b$ and $H_1{}^b$ are the restrictions of H^b to either end of the interval $\Delta[1]$, and $H^{\partial b}$ is its restriction to $\dot{\Delta}[n] \times \Delta[1]$. $H_0{}^{\partial b}$ and $H_1{}^{\partial b}$ then have an obvious meaning. There is a diagram

$$\begin{array}{ccccc} \mathbf{C}(H_0{}^b, H_0{}^{\partial b}) & \to & \mathbf{C}(H^b, H^b) & \leftarrow & \mathbf{C}(H_1{}^b, H_1{}^{\partial b}) \\ \| & & & & \| \\ \mathbf{C}(b, \partial b) & & & & \mathbf{C}(\Sigma^n, *). \end{array}$$

By [2], this becomes a diagram of weak homotopy equivalences when $\operatorname{Tot}_s(A \underline{\otimes} -)$ is applied, since in each codimension it is already a diagram of weak homotopy equivalences. This proves (5.1).

Calling (5.1) a "local triviality" result is justified by the fact that $\mathbf{C}(\Sigma^n, *)$ is a product as a cosimplicial space:

$$\mathbf{C}(\Sigma^n, *) = \mathbf{F}^+ \wedge S^n$$

where $\mathbf{F}^+$ is the cosimplicial space $\mathbf{F}$ with a disjoint basepoint adjoined in each codimension. The following *ad hoc* "Kunneth" formula determines the "homology" of this product object.

5.2. *Let* $\mathbf{C}$ *be a cosimplicial space with the property, in terms of* [2], *that the "*E^1 *term of its homology spectral sequence vanishes below the diagonal." Then there is a natural isomorphism*

$$\begin{aligned} \pi_i \operatorname{Tot}_s(A \underline{\otimes} (\mathbf{C}^+ \wedge S^n)) &= \pi_{i-n} \operatorname{Tot}_s(A \underline{\otimes} \mathbf{C}^+) \qquad & i \geq n \\ &= 0 & i < n. \end{aligned}$$

Note that the Eilenberg–Moore object $\mathbf{F}$ can be shown to have the quoted property whenever B is connected. For the proof of (5.2), proceed by induction on n. The statement is obviously true for $n = 0$, as long as S^0 is taken to be the disjoint union of two points. For $n > 0$ it is possible to choose simplicial representatives of the spaces involved so that there is a cofibration sequence

$$S^{n-1} \to D^n \to S^n$$

where D^n is some pointed contractible space. This gives for each k another cofibration sequence

$$\mathbf{C}^{k^+} \wedge S^{n-1} \to \mathbf{C}^{k^+} \wedge D^n \to \mathbf{C}^{k^+} \wedge S^n$$

and therefore an exact sequence of cosimplicial simplicial groups

$$A \underline{\otimes} \mathbf{C}^+ \wedge S^{n-1} \to A \underline{\otimes} \mathbf{C}^+ \wedge D^n \to A \underline{\otimes} \mathbf{C}^+ \wedge S^n.$$

By [2], this becomes a fibration sequence when $\operatorname{Tot}_s(-)$ is applied. The middle space in this fibration sequence clearly has vanishing homotopy groups, and the "E^1" hypothesis ensures that the base space is connected. An application of the fibration long exact homotopy sequence thus carries the induction one step further and completes the proof of (5.2).

By definition, the spectral sequence E_s^r comes from the filtration of $\operatorname{Tot}_s(A \otimes \mathbf{E})$ given by

$$\operatorname{Tot}_s(A \otimes \mathbf{E}(0)) \to \operatorname{Tot}_s(A \otimes \mathbf{E}(1)) \to \cdots \to \operatorname{Tot}_s(A \otimes \mathbf{E})$$

so that

$$E_s^1(i, j) = \pi_{i+j} \operatorname{Tot}_s((A \otimes \mathbf{E}(i))/A \otimes \mathbf{E}(i-1)).$$

There is, however, an alternative computation

$$E_s^1(i, j) = \pi_{i+j} \operatorname{Tot}_s(A \underline{\otimes} (\mathbf{E}(i)/\mathbf{E}(i-1))$$

which follows from the fact that $\operatorname{Tot}_s(-)$ transforms exact sequences of cosimplicial simplicial groups into fibration sequences. A slight dimension zero problem which arises in this computation can be resolved by noticing that $\operatorname{Tot}_s(A \otimes (\mathbf{E}(i)/\mathbf{E}(i-1)))$ is connected for $i > 0$, as a consequence of (5.1), (5.2), and the useful geometric decomposition:

$$\mathbf{E}(i)/\mathbf{E}(i-1) = \bigvee_{\substack{b \in B_i \\ b \text{ non-degenerate.}}} \mathbf{C}(b, \partial b)$$

Thus

$$A \underline{\otimes} (\mathbf{E}(i)/\mathbf{E}(i-1)) = \bigoplus_{\substack{b \in B_i \\ b \text{ non-degenerate}}} \mathbf{A} \underline{\otimes} \mathbf{C}(b, \partial b)$$

By (5.1), this has the same homotopy groups as

$$\bigoplus_{\substack{b \in B_i \\ b \text{ non-degenerate}}} A \underline{\otimes} (\mathbf{F}^+ \wedge S^i)$$

so, by (5.2)

$$E_s^{\,1}(i, j) = \bigoplus_{\substack{b \in B_i \\ b \text{ non-degenerate}}} \pi_j \operatorname{Tot}_s(A \otimes \mathbf{F})$$

$$= \mathbf{C}_i(B, \pi_j \operatorname{Tot}_s(A \otimes \mathbf{F})).$$

This concludes the computation of $E_s^{\,1}$.

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