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Twisted homological stability for general linear groups

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1. Introduction

Suppose that R is a principal ideal domain and that, for each n, ρ_n is a module over the general linear group $\mathrm{GL}_n(R)$. The major purpose of this paper is to show that for very general choices of $\{\rho_n\}$ the homology groups $H_k(\mathrm{GL}_n(R), \rho_n)$ *stabilize with respect to* n, i.e., for each fixed k assume a constant value as n becomes large. A secondary purpose is to show that this stability phenomenon has interesting consequences for Waldhausen's "algebraic K-theory of topological spaces" and ultimately for geometric topology.

The main stability theorem (Theorem 2.2) is explained in detail at the beginning of Section 2; it has an inductive statement too elaborate to summarize here. A sample application of 2.2 is the following. Let Ab be the category of abelian groups, and let $\mathrm{Ad}_n(R)$ be the abelian group of $n \times n$ matrices over R, considered as a $\mathrm{GL}_n(R)$-module by conjugation. Note that if T: Ab $\to$ Ab is a functor, there is a natural action of $\mathrm{GL}_n(R)$ on $T(\mathrm{Ad}_n(R))$.

1.1. PROPOSITION. *If T: Ab $\to$ Ab is a functor of finite degree (see* § 3 *and* [3]) *then the homology groups $H_k(\mathrm{GL}_n(R), T(\mathrm{Ad}_n(R)))$ stabilize with respect to n.*

Remark. Any tensor power, symmetric power, or exterior power functor is of finite degree.

To any space X, Waldhausen associates another space $A(X)$ whose homotopy groups are called the algebraic K-groups of X [10]. The space $A(X)$ is realizable as a Quillen plus-construction in a way that can be partially analyzed using the case $R = \mathbf{Z}$ of 1.1. This leads to

1.2. PROPOSITION. *Suppose that X is simply connected and that the homotopy groups of X are finitely generated. Then the homotopy groups of $A(X)$ are finitely generated.*

If the machinery of [4], [5] and [6] (cf. [11] and [12]) goes through as

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expected, Proposition 1.2 will lead to the following corollary.

1.3. PROPOSITION. *Suppose that M^n is a simply connected closed smooth n-manifold and that* $\mathrm{Diff}(M^n)$ *is the space of diffeomorphisms of M with the Whitney C^∞ topology. Then for $1 \leqq i \ll n$ the homotopy groups $\pi_i \mathrm{Diff}(M^n)$ are finitely generated.*

Remark. What "$i \ll n$" means depends upon what is known about the stable range for the smooth pseudo-isotopy space. In [4] "$i \ll n$" means "$i < n/6 - 7$".

Relationship to earlier work. This paper grew out of an attempt to understand the work of R. Charney [2] on stability properties of the trivial coefficient homology groups $H_k(\mathrm{GL}_n(\Lambda); \mathbf{Z})$, Λ a Dedekind domain. The key arguments in Section 2 are based on hers. For convenience, Section 2 deals only with principal ideal domains, but it seems likely that some form of Theorem 2.2 can be proved for Dedekind domains or even (see [7]) for more general rings.

Notation and terminology. Throughout the paper R will stand for a fixed principal ideal domain and $\mathrm{GL}_n = \mathrm{GL}_n(R)$ for the group of invertible $n \times n$ matrices over R. If G is a group and A is a right G-module, B a left G-module, then $A \otimes_G B$ and $\mathrm{Tor}_*^G(A, B)$ are sometimes used to denote the corresponding tensor product or Tor over the integral group ring of G. In general, all unspecified modules are left modules, so that $H_*(G, -) = \mathrm{Tor}_*^G(\mathbf{Z}, -)$, where $\mathbf{Z}$ is a trivial right module.

Organization of the paper. Section 2 contains a proof of the main stability theorem, Section 3 gives the proof of 1.1, and the final section makes the application of 1.1 to Waldhausen's theory.

2. Homological Stability

For any n let $I: \mathrm{GL}_n \to \mathrm{GL}_{n+1}$ denote the "upper" inclusion, i.e., the map which embeds GL_n as the subgroup of GL_{n+1} consisting of the matrices

$$\{(r_{ij}) \mid r_{ij} = \delta_{ij} \text{ (Kronecker delta) if } i = n + 1 \text{ or } j = n + 1\} .$$

A *coefficient system* ρ is a sequence of GL_n-modules ρ_n together with GL_n-maps $F_n: \rho_n \to I^*(\rho_{n+1})$. (Here $I^*(\rho_{n+1})$ is the GL_n-module obtained by restricting the GL_{n+1}-module ρ_{n+1} to GL_n via I.) A *map* $\tau: \rho \to \sigma$ of coefficient systems is a collection of GL_n-maps $\tau_n: \rho_n \to \sigma_n$ such that the diagrams

$$\begin{array}{ccc} \rho_n & \longrightarrow & I^*\rho_{n+1} \\ {\scriptstyle \tau_n}\big\downarrow & & \big\downarrow{\scriptstyle I^*\tau_{n+1}} \\ \sigma_n & \longrightarrow & I^*\sigma_{n+1} \end{array}$$

commute. The kernel and cokernel of τ are defined in the obvious way; these are again coefficient systems.

Any coefficient system ρ gives rise for each integer k to an inductive system of homology groups

$$\longrightarrow H_k(\mathrm{GL}_n; \rho_n) \longrightarrow H_k(\mathrm{GL}_{n+1}; \rho_{n+1}) \longrightarrow \cdots .$$

The coefficient system is said to be *stable* if for any given k this inductive system is eventually constant.

The purpose of this section is to give a general way of recognizing stable coefficient systems. The main theorem will be stated after some definitions and constructions.

Let $J\colon \mathrm{GL}_n \to \mathrm{GL}_{n+1}$ denote the "lower" inclusion, i.e., the map which embeds GL_n as the subgroup of GL_{n+1} consisting of the matrices $\{(r_{ij}) \mid r_{ij} = \delta_{ij}$ if $i = 1$ or $j = 1\}$. If ρ is any coefficient system, define the "shifted" coefficient system $\Sigma\rho$ by letting $(\Sigma\rho)_n$ equal $J^*(\rho_{n+1})$ and using the obvious structure maps. Let $\Sigma^0\rho = \rho$ and, for any positive integer j, let $\Sigma^j\rho$ be defined inductively by $\Sigma^j\rho = \Sigma\Sigma^{j-1}\rho$. The coefficient system ρ is said to be *strongly stable* if $\Sigma^j\rho$ is stable for each $j \geqq 0$.

For each $n \geqq 2$ let s_n denote the permutation matrix in GL_n which interchanges the last two standard basis elements of R^n under the standard left action of GL_n on the module R^n of column vectors. The coefficient system ρ is said to be *central* if for each n the matrix s_{n+2} acts trivially on the image of the composite map $F_{n+1}F_n\colon \rho_n \to \rho_{n+2}$. Note that $F_{n+1}F_n$ is only a map of GL_n-modules; nevertheless its image is a subgroup of the GL_{n+2}-module ρ_{n+2}, and so it is meaningful to ask that the matrix s_{n+2} leave this image pointwise fixed.

For $n \geqq 2$ let $c_n \in \mathrm{GL}_n$ be the cyclic permutation matrix with $c_n(e_i) = e_{i+1}$ $(i < n)$, $c_n e_n = e_1$, where e_i is the i^{th} standard basis element of R^n. It is clear that for each $g \in \mathrm{GL}_n$ there is a matrix equality

$$I(g) = c_{n+1}^{-1} J(g) c_{n+1}$$

so that if M is a GL_{n+1}-module, the action $\mu(c_{n+1})$ of c_{n+1} on M gives a GL_n-map

$$\mu(c_{n+1})\colon I^*(M) \longrightarrow J^*(M) .$$

This implies that if ρ is a coefficient system there are composite GL_n-maps

$$\tau_n\colon \rho_n \xrightarrow{F_n} I^*(\rho_{n+1}) \xrightarrow{\mu(c_{n+1})} J^*(\rho_{n+1}) .$$

2.1. LEMMA. *If ρ is a central coefficient system the above homomorphisms $\tau_n\colon \rho_n \to J^*(\rho_{n+1})$ are the constituents of a coefficient system map $\tau\colon \rho \to \Sigma\rho$.*

This lemma will be proved below. The main result of this section is the following one.

2.2. THEOREM. *Suppose that* ρ *is a central coefficient system and that* $\tau\colon \rho \to \Sigma\rho$ *is the map of* 2.1. *Then if* kernel (τ) *and* cokernel (τ) *are strongly stable, so is* ρ.

Example. If ρ is a constant coefficient system, then τ is an isomorphism; in this case the theorem specializes to part of Charney's theorem [2]. There are many other examples in Section 3.

Proof of Lemma 2.1. It is necessary to check that for each n the diagram

$$\begin{array}{ccc} \rho_n & \xrightarrow{F_n} & \rho_{n+1} \\ {\scriptstyle \mu(c_{n+1})F_n}\downarrow & & \downarrow{\scriptstyle \mu(c_{n+2})F_{n+1}} \\ \rho_{n+1} & \xrightarrow{F_{n+1}} & \rho_{n+2} \end{array}$$

commutes. Since F_{n+1} is a GL_{n+1}-map, $\rho_{n+1} \to I^*(\rho_{n+2})$, there is an equality $F_{n+1} \circ \mu(c_{n+1}) = \mu(Ic_{n+1}) \circ F_{n+1}$. Commutativity follows from the matrix equation $c_{n+2} = I(c_{n+1}) \cdot s_{n+2}$ and the fact that ρ is central.

The rest of this section is occupied with the proof of 2.2. The first step is to set up some homological machinery. Suppose that $h\colon G_1 \to G_2$ is a map of groups, M_i is a module over G_i $(i = 1, 2)$, and $F\colon M_1 \to h^*(M_2)$ is a G_1-module map. (From now on, unless there is a possibility for confusion, we will leave out the h^* and write $F\colon M_1 \to M_2$.) Suppose that S is a fixed *right* G_2-module which is considered to be a G_1-module via h. Let $R_i \to S$ be a projective resolution of S as a G_i-module $(i = 1, 2)$. By standard homological algebra there is a G_1-chain-map $R_1 \to R_2$, unique up to homotopy, that covers the identity map $S \to S$. The *mapping cone* [8, p. 46] of the induced map $R_1 \otimes_{G_1} M_1 \to R_2 \otimes_{G_2} M_2$ is a chain complex whose homology groups are denoted $\mathrm{Tor}_i^{(G_2, G_1)}(S; M_2, M_1)$. These groups are functorial in G_1, G_2, M_1, M_2 and S; if S is the trivial module $\mathbf{Z}$, they are also written $H_i(G_2, G_1; M_2, M_1)$.

The following properties of these "doubly relative" groups can be derived in a straightforward way. To simplify the statements, let $\mathrm{Rel}_i(M_2, M_1)$ temporarily denote $\mathrm{Tor}_i^{(G_2, G_1)}(S; M_2, M_1)$.

2.3. LEMMA. *There is a long exact sequence*

$$\longrightarrow \mathrm{Tor}_i^{G_1}(S, M_1) \longrightarrow \mathrm{Tor}_i^{G_2}(S, M_2) \longrightarrow \mathrm{Rel}_i(M_2, M_1) \longrightarrow \mathrm{Tor}_{i-1}^{G_1}(S, M_1) \longrightarrow .$$

2.4. LEMMA. *Suppose that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1' & \longrightarrow & M_1 & \longrightarrow & M_1'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_2' & \longrightarrow & M_2 & \longrightarrow & M_2'' & \longrightarrow & 0 \end{array}$$

is a map of short exact sequences, where the upper sequence is a sequence of G_1-modules, the lower is a sequence of G_2-modules, and the vertical maps are G_1-maps. Then there is a long exact sequence

$$\to \mathrm{Rel}_i(M_2', M_1') \to \mathrm{Rel}_i(M_2, M_1) \to \mathrm{Rel}_i(M_2'', M_1'') \to \mathrm{Rel}_{i-1}(M_2'', M_1') \to .$$

2.5. LEMMA. *Suppose that $G_1 \to G$, $G_2 \to G$ are surjections with kernels K_1, K_2 respectively, such that the diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{h} & G_2 \\ & \searrow \swarrow & \\ & G & \end{array}$$

commutes. Suppose in addition that the G_2-module structure on S factors through a G-module structure. Then there is a first-quadrant homology spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^G_p\big(S, H_q(K_2, K_1; M_2, M_1)\big) \Rightarrow \mathrm{Rel}_{p+q}(M_2, M_1) .$$

2.6. COROLLARY. *In the situation of the previous lemma, suppose that $H_q(K_2, K_1; M_2, M_1) = 0$ for $q < k$. Then $\mathrm{Rel}_i(M_2, M_1) = 0$ for $i < k$ and there is a natural isomorphism*

$$\mathrm{Rel}_k(M_2, M_1) \approx S \otimes_G H_k(K_2, K_1; M_2, M_1) .$$

If ρ is a coefficient system, let $\mathrm{Rel}^n_i(\rho)$ denote the group $H_i(\mathrm{GL}_{n+1}, \mathrm{GL}_n; \rho_{n+1}, \rho_n)$; this is of course computed with respect to the upper inclusion $I: \mathrm{GL}_n \to \mathrm{GL}_{n+1}$ and the coefficient system structure map $\rho_n \to I^*(\rho_{n+1})$. By Lemma 2.3, ρ is stable if and only if for any i the groups $\mathrm{Rel}^n_i(\rho)$ vanish for all sufficiently large n. We will now set up a spectral sequence that converges to these relative groups.

Pick some $n \geqq 3$. For $i \leqq n$, let P_i denote the following subgroup of GL_n:

$$P_i = \begin{pmatrix} [\mathrm{GL}_i] & 0 \\ 0 & [\mathrm{GL}_{n-i}] \end{pmatrix} ,$$

and for $i \leqq n + 1$ let P_i' be the corresponding subgroup of GL_{n+1}:

$$P_i' = \begin{pmatrix} [\mathrm{GL}_i] & 0 \\ 0 & [\mathrm{GL}_{n-i+1}] \end{pmatrix} .$$

Let S_i be Charney's i^{th} *split Steinberg module* (written $[R^i]$ in the notation of [2]) considered as a *right* GL_i-module; note that S_1 is the trivial GL_1-module Z. Consider S_i to be a module over P_i and P_i' via the projections $P_i \to \mathrm{GL}_i$, $P_i' \to \mathrm{GL}_i$.

Recall briefly from [2] how the modules S_i $(i \geqq 2)$ are defined. Let R^i

denote the i-fold direct sum of R with itself, considered as a left R-module. A *splitting* α of R^i is an ordered pair (P, Q) of submodules of R^i such that $P \neq 0$, $Q \neq 0$, $P \cap Q = 0$, and $P + Q = R^i$. If $\alpha_1 = (P_1, Q_1)$ and $\alpha_2 = (P_2, Q_2)$ are two such splittings, write $\alpha_1 < \alpha_2$ if $P_1 \subseteqq P_2$, $Q_1 \supseteqq Q_2$ and $\alpha_1 \neq \alpha_2$. The *split building* of R^i, written $\mathrm{SB}[R^i]$, is the $(i-2)$-dimensional simplicial complex whose k simplices, for $k \leqq i-2$, are given by ordered $(k+1)$-tuples $(\alpha_0, \alpha_1, \cdots, \alpha_k)$ of splittings of R^i such that $\alpha_0 < \alpha_1 < \cdots < \alpha_k$. The group GL_i acts naturally on $\mathrm{SB}[R^i]$ and thus, for each $k \geqq 0$, on the reduced integral homology group $\tilde{H}_k\,\mathrm{SB}[R^i]$. For $k \neq i-2$, this homology group vanishes; for $k = i-2$, it is denoted S_i and called the i^{th} split Steinberg module of R.

If $\alpha = (P, Q)$ is a splitting of R^i, let $rk(\alpha)$ denote the rank of P as an R-module. For each $l \geqq 1$ there is a subcomplex of $\mathrm{SB}[R^i]$ spanned by the simplices $(\alpha_0, \cdots, \alpha_k)$ such that $rk(\alpha_k) \leqq l$. It is the homology spectral sequence of this finite filtration of $\mathrm{SB}[R^i]$ that degenerates into the long exact sequence appearing in the proof of the following lemma.

2.7. LEMMA. *There is a first quadrant spectral sequence*

$$E^1_{i,j} = \mathrm{Tor}_j^{(P'_{i+1}, P_{i+1})}(S_{i+1}; \rho_{n+1}, \rho_n) \Longrightarrow \mathrm{Rel}^n_{i+j}(\rho)\ .$$

Remarks. The differential d_r in the spectral sequence has bi-degree $(-r, r-1)$. The relative groups in the E^1-term are computed with respect to the upper inclusion $I\colon P_{i+1} \to P'_{i+1}$; for large i, the formula for E^1 is to be interpreted as $E^1_{i,j} = 0$ $(i > n)$, $E^1_{n,j} = \mathrm{Tor}_j^{\mathrm{GL}_{n+1}}(S_{n+1}, \rho_{n+1})$. The edge homomorphism

$$H_*(P'_1, P_1; \rho_{n+1}, \rho_n) = E^1_{0,*} \longrightarrow \mathrm{Rel}^n_*(\rho) = H_*(\mathrm{GL}_{n+1}, \mathrm{GL}_n; \rho_{n+1}, \rho_n)$$

is induced by the natural inclusion of pairs $(P'_1, P_1) \to (\mathrm{GL}_{n+1}, \mathrm{GL}_n)$.

Proof of 2.7. Let Λ, Λ' be the integral group rings of GL_n, GL_{n+1}, respectively. According to [2] the upper inclusion $I\colon \mathrm{GL}_n \to \mathrm{GL}_{n+1}$ induces a map of exact sequences

$$\begin{array}{ccccccccccc}
0 \longleftarrow & \mathbf{Z} & \longleftarrow & S_1 \otimes_{P_1} \Lambda & \longleftarrow & S_2 \otimes_{P_2} \Lambda & \longleftarrow \cdots \longleftarrow & S_n & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longleftarrow & \mathbf{Z} & \longleftarrow & S_1 \otimes_{P'_1} \Lambda' & \longleftarrow & S_2 \otimes_{P'_2} \Lambda' & \longleftarrow \qquad \longleftarrow & S_n \otimes_{P'_n} \Lambda' & \longleftarrow \ .
\end{array}$$

Here the upper sequence is an exact sequence of right GL_n-modules, the lower is an exact sequence of right GL_{n+1}-modules, and the vertical maps are GL_n-maps. In fact, the vertical map

$$S_i \otimes_{P_i} \Lambda \longrightarrow S_i \otimes_{P'_i} \Lambda'$$

is constructed in the obvious way using the inclusion $P_i = \mathrm{GL}_i \times \mathrm{GL}_{n-i} \to \mathrm{GL}_i \times \mathrm{GL}_{n-i+1} = P'_i$ which is the product of the identity map $\mathrm{GL}_i \to \mathrm{GL}_i$ and the upper inclusion $\mathrm{GL}_{n-i} \to \mathrm{GL}_{n-i+1}$.

Let $X = \{X_i\}$ be the chain complex

$$S_1 \otimes_{P_1} \Lambda \longleftarrow S_2 \otimes_{P_2} \Lambda \longleftarrow \cdots \longleftarrow S_n \otimes_{P_n} \Lambda$$

of right GL_n-modules, and let X' be the corresponding chain complex of right GL_{n+1}-modules. Choose a double complex $Y = \{Y_{i,j}\}$ of right GL_n-modules which is a projective resolution of X in the sense of [1, p. 363]; in particular, then, for $0 \leqq i \leqq n-1$ the single complex $Y_{i,*}$ is a projective resolution of $X_i = S_{i+1} \otimes_{P_{i+1}} \Lambda$. Let $Y' = \{Y'_{i,j}\}$ be a corresponding resolution of X' over GL_{n+1}. Standard methods give an essentially unique GL_n-map $Y \to Y'$ of double complexes which covers the given map $X \to X'$.

For each $i \geqq 0$, let $C_{i,*}$ be the mapping cone [8, p. 46] of the induced chain complex map

$$Y_{i,*} \otimes_{\mathrm{GL}_n} \rho_n \longrightarrow Y'_{i,*} \otimes_{\mathrm{GL}_{n+1}} \rho_{n+1} .$$

The modules $C_{i,j}$ form a double complex C. Since X and X' are resolutions of the trivial module $\mathbf{Z}$ over GL_n and GL_{n+1} respectively, it follows that the homology of the total complex of C is $\mathrm{Rel}^n_*(\rho)$. Filtering C according of its first subscript gives the spectral sequence of 2.7; the E^1-term can be identified by Shapiro's lemma.

2.8. LEMMA. *Let ρ be a coefficient system that satisfies the hypotheses of* 2.2. *Then for any $i \geqq 0$, and $m \geqq 0$ and all sufficiently large n, the composite map $\Sigma^{m-1}\tau \circ \Sigma^{m-2}\tau \circ \cdots \circ \Sigma\tau \circ \tau$ induces isomorphisms* $\mathrm{Rel}^n_i(\rho) \approx \mathrm{Rel}^n_i(\Sigma^m \rho)$.

Proof. Let σ_1 and σ_3 be the kernel and cokernel respectively of the map $\tau: \rho \to \Sigma\rho$, and let σ_2 be the image of τ. By 2.4, the coefficient system short exact sequences

$$0 \longrightarrow \Sigma^m\sigma_1 \longrightarrow \Sigma^m\rho \longrightarrow \Sigma^m\sigma_2 \longrightarrow 0 ,$$
$$0 \longrightarrow \Sigma^m\sigma_2 \longrightarrow \Sigma^{m+1}\rho \longrightarrow \Sigma^m\sigma_3 \longrightarrow 0 ,$$

give rise to long exact sequences when Rel^n_* is applied. Given these sequences and the fact that σ_1 and σ_3 are strongly stable, the lemma can be proved by induction on m; i.e., by showing that for sufficiently large n there are isomorphisms $\mathrm{Rel}^n_i(\Sigma^m\sigma_2) \approx \mathrm{Rel}(\Sigma^{m+1}\rho)$.

Proof of 2.2. Suppose by induction on k that for any $i < k$ the groups $\mathrm{Rel}^n_i(\rho)$ are known to vanish for all sufficiently large n. (This is trivial for $k = 0$.) We will show that $\mathrm{Rel}^n_k(\rho)$ vanishes for all sufficiently large n, which, by 2.8, is enough to prove that ρ is strongly stable.

Using 2.8 and the induction hypothesis, pick an integer N such that

$\mathrm{Rel}_j^n(\Sigma^m\rho) = 0$ for all $j < k$, $m \leqq k+1$, $n \geqq N-k-1$. Pick $n \geqq N$, and let P_i', P_i be the subgroups of GL_{n+1}, GL_n, respectively, that appear in Lemma 2.7. The upper inclusion $I\colon \mathrm{GL}_n \to \mathrm{GL}_{n+1}$ induces a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{GL}_{n-i} & \longrightarrow & P_i & \longrightarrow & \mathrm{GL}_i & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{GL}_{n-i+1} & \longrightarrow & P_i' & \longrightarrow & \mathrm{GL}_i & \longrightarrow & 1 \end{array}$$

of short exact sequences, in which the kernel map $\mathrm{GL}_{n-i} \to \mathrm{GL}_{n-i+1}$ is also the upper inclusion. The restriction of ρ_n to the above copy of GL_{n-i} in GL_n is $(\Sigma^i\rho)_{n-i}$, and the corresponding restriction of ρ_{n+1} to the above copy of GL_{n-i+1} in GL_{n+1} is $(\Sigma^i\rho)_{n-i+1}$. It follows from the choice of n that the relative group

$$\begin{array}{c} H_j\big(\mathrm{GL}_{n-i+1}, \mathrm{GL}_{n-i}; (\Sigma^i\rho)_{n-i+1}, (\Sigma^i\rho)_{n-i}\big) \\ \| \\ \mathrm{Rel}_j^{n-i}(\Sigma^i\rho) \end{array}$$

vanishes if $j < k$ and $i \leqq k+1$, so, if $E_{i,j}^n$ is the spectral sequence of 2.7, Lemma 2.6 implies that $E_{i,j}^1 = 0$ if $j < k$ and $i \leqq k$. This gives a surjection

$$H_0\big(\mathrm{GL}_1; \mathrm{Rel}_k^{n-1}(\Sigma\rho)\big) = E_{0,k}^1 \twoheadrightarrow \mathrm{Rel}_k^n(\rho)\ .$$

Using 2.8 to increase N, if necessary, so that for $n \geqq N$ the map τ induces an isomorphism $\mathrm{Rel}_k^{n-1}(\rho) \approx \mathrm{Rel}_k^{n-1}(\Sigma\rho)$, we conclude that for $n \geqq N$ there is a composite epimorphism

$$\mathrm{Rel}_k^{n-1}(\rho) \longrightarrow \mathrm{Rel}_k^{n-1}(\Sigma\rho) \longrightarrow E_{0,k}^1 \longrightarrow \mathrm{Rel}_k^n(\rho)\ .$$

Unravelling the above constructions shows that this epimorphism is induced by the commutative diagrams

$$\begin{array}{ccc} \mathrm{GL}_{n-1} & \xrightarrow{I} & \mathrm{GL}_n \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathrm{GL}_n & \xrightarrow[I]{} & \mathrm{GL}_{n+1}\ , \end{array}$$

$$\begin{array}{ccc} \rho_{n-1} & \xrightarrow{F_{n-1}} & \rho_n \\ {\scriptstyle \mu(c_n)F_{n-1}}\downarrow & & \downarrow{\scriptstyle \mu(c_{n+1})F_n} \\ \rho_n & \xrightarrow[F_n]{} & \rho_{n+1}\ . \end{array}$$

Call this epimorphism g.

Consider the map of long exact sequences

$$\begin{array}{ccccccc} \longrightarrow H_k(\mathrm{GL}_n, \rho_n) & \xrightarrow{a_1} & \mathrm{Rel}_k^{n-1}(\rho) & \xrightarrow{a_2} & H_{k-1}(\mathrm{GL}_{n-1}, \rho_{n-1}) \longrightarrow \\ \updownarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle l_2} \\ \longrightarrow H_k(\mathrm{GL}_{n+1}, \rho_{n+1}) & \xrightarrow{b_1} & \mathrm{Rel}_k^n(\rho) & \xrightarrow{b_2} & H_{k-1}(\mathrm{GL}_n, \rho_n) \longrightarrow \end{array}$$

where l_1 is induced by the pair $(J, \mu(c_{n+1})F_n)$ and l_2 by the pair $(J, \mu(c_n)F_{n-1})$. There is another obvious map between the same pair of long exact sequences

$$\begin{array}{ccccccc} \longrightarrow H_k(\mathrm{GL}_n, \rho_n) & \xrightarrow{a_1} & \mathrm{Rel}_k^{n-1}(\rho) & \xrightarrow{a_2} & H_{k-1}(\mathrm{GL}_{n-1}, \rho_{n-1}) \longrightarrow \\ \downarrow l_1' & & \downarrow g' & & \downarrow l_2' \\ \longrightarrow H_k(\mathrm{GL}_{n+1}, \rho_{n+1}) & \xrightarrow{b_1} & \mathrm{Rel}_k^n(\rho) & \xrightarrow{b_2} & H_{k-1}(\mathrm{GL}_n, \rho_n) \longrightarrow , \end{array}$$

where l_1' is induced by (I, F_n) and l_2' by (I, F_{n-1}). The map g' is identically zero, because it factors through the trivial group $H_k(\mathrm{GL}_n, \mathrm{GL}_n; \rho_n, \rho_n)$. Since c_n and c_{n+1} conjugate the appropriate upper inclusions to the corresponding lower inclusions, it follows from standard homological algebra that the maps l_1, l_2 are identical with l_1', l_2' respectively. Consequently, $b_2 g = l_2 a_2 = l_2' a_2 = b_2 g' = 0$; since g is surjective, this implies that b_2 is zero and thus that b_1 is surjective. Choosing $n \geqq N + 1$, we can assume that a_1 is surjective. However, $ga_1 = b_1 l_1 = b_1 l_1' = g' a_1 = 0$; since a_1 is surjective, this implies that g is zero, and hence, since g is surjective, that $\mathrm{Rel}_k^n(\rho)$ is zero. This finishes the inductive step and completes the proof.

3. Examples of strongly stable coefficient systems

Let A and B be abelian categories, and let T: A $\to$ B be a functor. Under the assumption that $T(0) = 0$, Eilenberg and MacLane define the *cross-effects* T_k $(k \geqq 1)$ of T; these are functors

$$T_k\colon \mathrm{A}^k \longrightarrow \mathrm{B}$$

which measure the deviation of T from additivity [3, §9]. Then inductive construction of the T_k given by [3, Lemma 9.9] is meaningful for arbitrary functors if $T_1(X)$ is taken to be the kernel of the natural split epimorphism $T(X) \to T(0)$; with this convention the fundamental equations characterizing cross effects [3, Theorem 9.1] remain valid after the addition of an extra term $T(0)$ to the right hand side. For instance, given X, $Y \in$ A,

$$T(X \oplus Y) = T(0) \oplus T_1(X) \oplus T_1(Y) \oplus T_2(X, Y) .$$

The functor T is said to be of *degree zero* if $T_1 = 0$, i.e., if T is constant; in general T is said to be of *degree* $k \geqq 0$ if $T_k \neq 0$ but $T_{k+1} = 0$. The functor T has *finite degree* if it is of degree k for some $k \geqq 0$.

Let λ be the natural coefficient system which assigns to each group GL_n its standard module R^n—the elements of λ_n are the column vectors of length n on which GL_n acts by matrix multiplication from the left. Let $\bar{\lambda}$ be the corresponding contragredient coefficient system, so that the elements of $\bar{\lambda}_n$ are column vectors of length n on which GL_n acts by inverse-transpose matrix multiplication from the left. Let R-Mod be the category of R-modules and Ab the category of abelian groups. If $T: (R\text{-Mod})^2 \to \text{Ab}$ is a functor, there is a natural coefficient system $T(\lambda, \bar{\lambda})$ defined by setting

$$T(\lambda, \bar{\lambda})_n = T(\lambda_n, \bar{\lambda}_n)$$

with the obvious diagonal GL_n-action.

3.1. LEMMA. *If $T: (R\text{-Mod})^2 \to \text{Ab}$ is a functor of finite degree then the coefficient system $T(\lambda, \bar{\lambda})$ is strongly stable.*

Proof. The proof is by induction on the degree of T. If T has degree zero, then $T(\lambda, \bar{\lambda})$ is a constant coefficient system and the lemma follows directly from 2.2 or from Charney's theorem.

Suppose that the lemma is known to be true for all functors of degree $<k$ ($k \geqq 1$) and that T has degree k. Since λ and $\bar{\lambda}$ are central, $T(\lambda, \bar{\lambda})$ is also central. Let R denote the trivial constant coefficient system. For each of the coefficient systems λ, $\bar{\lambda}$, the map τ of 2.1 is the inclusion of the first factor in a direct sum decomposition

$$\Sigma\lambda = \lambda \oplus R\,,$$
$$\Sigma\bar{\lambda} = \bar{\lambda} \oplus R\,.$$

It follows that for $T(\lambda, \bar{\lambda})$, the map τ is the inclusion of the first factor in the corresponding decomposition

$$\Sigma T(\lambda, \bar{\lambda}) = T(\lambda, \bar{\lambda}) \oplus T_1(R, R) \oplus T_2(\lambda, \bar{\lambda}, R, R)\,.$$

It follows from [3, Lemma 9.9] that the functor $T_2(-, -, R, R): (R\text{-Mod})^2 \to \text{Ab}$ has degree less than k, and so by induction the coefficient system $T_2(\lambda, \bar{\lambda}, R, R)$ is strongly stable. The coefficient system $T_1(R, R)$ is constant and therefore likewise strongly stable. Strong stability of $T(\lambda, \bar{\lambda})$ follows from 2.2.

3.2. LEMMA. *Let* A, B, C *be abelian categories and let $S: \mathrm{A} \to \mathrm{B}$ and $T: \mathrm{B} \to \mathrm{C}$ be functors. If S has degree k and T has degree l, then the composite TS has degree $\leqq k \cdot l$.*

The proof of this is left to the reader.

Proof of Proposition 1.1. The tensor product functor $S: (R\text{-Mod})^2 \to \text{Ab}$ given by $S(X, Y) = X \otimes_R Y$ has degree 2. Since $\mathrm{Ad}_n(R)$ is isomorphic as a GL_n-module to $S(\lambda, \bar{\lambda})$, the proposition follows from 3.1 and 3.2.

4. The Waldhausen space

The purpose of this section is to prove Proposition 1.2. Recall that $A(X)$ is an H-space, with component group $\pi_0 A(X)$ isomorphic to $\mathbf{Z}$. Since the identity component of $A(X)$ can be realized as a plus construction $\widehat{\mathrm{BGL}}(X)^+$ (see [10, §2], [13], and [14]), Proposition 1.2 follows from

4.1. LEMMA. *If X is as in* 1.2, *then the integral homology groups of* $\widehat{\mathrm{BGL}}(X)$ *are finitely generated.*

Let $\mathrm{GL}(\mathbf{Z}) = \varinjlim \mathrm{GL}_n(\mathbf{Z})$ be the infinite general linear group over $\mathbf{Z}$, let $\mathrm{Ad}(\mathbf{Z}) = \varinjlim \mathrm{Ad}_n(\mathbf{Z})$ be the additive group of infinite matrices over $\mathbf{Z}$, and let $\pi_k^s(\Omega X)$ be the k^{th} unreduced stable homotopy group of the loop space ΩX. (The direct limits in the definitions of $\mathrm{GL}(\mathbf{Z})$ and $\mathrm{Ad}(\mathbf{Z})$ are taken with respect to the relevant upper inclusion maps (see §2).) Let $\mathrm{GL}(\mathbf{Z})$ act trivially on $\pi_k^s(\Omega X)$ and act on $\mathrm{Ad}(\mathbf{Z})$ by conjugation. For our purposes the important properties of $\widehat{\mathrm{BGL}}(X)$ (which hold under the assumption that X is 1-connected) are

(i) $\pi_1(\widehat{\mathrm{BGL}}(x)) = \mathrm{GL}(\mathbf{Z})$, and

(ii) for $k > 1$, $\pi_k \widehat{\mathrm{BGL}}(x)$ is isomorphic as a $\mathrm{GL}(\mathbf{Z})$-module to the tensor product $\mathrm{Ad}(\mathbf{Z}) \otimes \pi_{k-1}^s(\Omega X)$.

The proof of 4.1 consists in showing by induction on k that the integral homology groups of the Postnikov stages $P_k\widehat{\mathrm{BGL}}(X)$ are finitely generated. This depends on a general algebraic result.

Let $\mathfrak{M}$ be the smallest class of $\mathrm{GL}(\mathbf{Z})$-modules which has the following properties:

(a) $\mathfrak{M}$ contains $\mathrm{Ad}(\mathbf{Z})$.

(b) $\mathfrak{M}$ contains all finitely generated abelian groups with trivial $\mathrm{GL}(\mathbf{Z})$-action.

(c) If $\mathfrak{M}$ contains the modules A and B, then, for $n \geqq 2$ and $j \geqq 0$, $\mathfrak{M}$ contains the module $H_j(K(A, n), B)$.

4.2. LEMMA. *If $A \in \mathfrak{M}$, then the groups $H_j(\mathrm{GL}(\mathbf{Z}); A)$ are finitely generated.*

This will be proved below.

Proof of 4.1. We will show by induction on k that for any integer k and module $A \in \mathfrak{M}$ the (twisted) homology groups $H_j(P_k\mathrm{BGL}(X); A)$ are finitely generated.

The case $k = 1$ is Lemma 4.2. Suppose that the inductive hypothesis is known for $l < k$ where $k > 1$. Pick $A \in \mathfrak{M}$. The Serre spectral sequence of the fibration

$$K(\pi_k\widehat{\mathrm{BGL}}(X), k) \longrightarrow P_k\widehat{\mathrm{BGL}}(X) \longrightarrow P_{k-1}\widehat{\mathrm{BGL}}(X)$$

has the form

$$E^2_{p,q} = H_p(P_{k-1}\widehat{\mathrm{BGL}}(X)); H_q(K(\mathrm{Ad}(\mathbf{Z}) \otimes \pi^s_{k-1}(\Omega X), k), A)$$
$$\Downarrow$$
$$H_{p+q} P_k \widehat{\mathrm{BGL}}(X) .$$

The $\mathrm{GL}(\mathbf{Z})$-modules $H_q(K(\mathrm{Ad}(\mathbf{Z}) \otimes \pi^s_{k-1}(\Omega X), k), A)$ all belong to $\mathfrak{M}$, so it follows by induction that all of the groups in this E^2-term are finitely generated. This completes the proof.

4.3. LEMMA. *If $n \geqq 1$ and $j \geqq 0$, the functor $T: \mathrm{Ab}^2 \to \mathrm{Ab}$ given by $T(A, B) = H_j(K(A, n), B)$ has degree $\leqq (j/n) + 1$.*

This is easy to prove using the fact that the reduced homology of a k-fold smash product of $(n-1)$-connected spaces vanishes below dimension nk (cf. [3, §20]).

Proof of 4.2. Any coefficient system ρ gives rise to a $\mathrm{GL}(\mathbf{Z})$-module $\lim(\rho)$ by the formula $\lim(\rho) = \varinjlim \rho_n$. Since homology commutes with direct limits, there is an isomorphism $H_*(\mathrm{GL}(\mathbf{Z}), \lim(\rho)) \approx \varinjlim H_*(\mathrm{GL}_n(\mathbf{Z}), \rho_n)$. It follows from [9, Cor. 3] that the groups $H_j(\mathrm{GL}_n(\mathbf{Z}), \rho_n)$ are finitely generated if ρ_n is a $\mathrm{GL}_n(\mathbf{Z})$-module which is finitely generated as an abelian group. Consequently, the groups $H_j(\mathrm{GL}(\mathbf{Z}), \lim(\rho))$ are finitely generated if ρ is stable and each ρ_n is finitely generated as an abelian group.

Taken together, Lemmas 4.3 and 3.2 imply that each of the modules A in the class $\mathfrak{M}$ can be expressed as $T(\mathrm{Ad}(\mathbf{Z}))$, where $T: \mathrm{Ab} \to \mathrm{Ab}$ is a functor of finite degree which commutes with direct limits and preserves the subcategory of finitely generated abelian groups. Let λ and $\bar{\lambda}$ be the coefficient systems of Section 3 in the case $R = \mathbf{Z}$, and let T be such a functor. Then $T(\mathrm{Ad}(\mathbf{Z})) = \lim T(\lambda \otimes \bar{\lambda})$, the coefficient system $T(\lambda \otimes \bar{\lambda})$ is stable by 1.1 (see §3), and each GL_n-module $T(\lambda_n \otimes \bar{\lambda}_n)$ is a finitely generated abelian group. The lemma follows at once.

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