ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, Nos. 2–3, 1978

A WHITEHEAD THEOREM FOR LONG TOWERS OF SPACES†

BY

E. DROR AND W. G. DWYER†

ABSTRACT

We show that one can construct the universal R-homology isomorphism $K \to E_R X$ of Bousfield [1] by a transfinite iteration of an elementary homology correction map. This correction map is essentially the same as the one used classically to define Adams spectral sequence. This yields a topological characterization of the class of local spaces as the smallest s containing $K(A, n)$'s and closed under homotopy inverse limit.

1. Introduction

Suppose that R is a subring of the rational numbers or a finite field of the form $\mathbf{Z}/p\mathbf{Z}$, p prime. In [1] Bousfield showed that any space X has a functorial *R-homology localization*; this is a space X_R together with a map $X \to X_R$ which is terminal, up to homotopy, in the category of all maps $X \to Y$ that induce isomorphisms on mod R homology. This paper proves a "Whitehead theorem" which is adapted to recognizing inverse limit constructions of X_R. In particular, the theorem shows that X_R can be obtained from X by transfinite iteration of an elementary homology approximation technique.

1.1. *Organization of the paper.* For background purposes, Section 2 describes in some detail our proposed construction for X_R. Section 3 contains some preliminary algebra, and Section 4 has a statement and proof of the Whitehead theorem itself. The last section shows how the Whitehead theorem can be used to demonstrate that the construction given in Section 2 actually works.

1.2. REMARK. Section 2 is a straightforward attempt to transpose the algebraic towers of [2, §3, §8] into geometry. A less direct but more sophisticated

† This research was partially supported by the National Science Foundation, Grant # MCS76–08795, and by the U.S.–Israel Bi-National Science Foundation Grant # 680.

Received December 14, 1976

inverse limit construction of X_R appears in [4]. Our work here ultimately depends on a small but essential collection of Bousfield's algebraic lemmas from [2, §1–2, §6–7].

1.3. *Notation and terminology.* The word *space* is used as a synonym for *simplicial set* ([5], [6]). The symbol R will always denote a fixed ring of the type described above; a space X is said to be *R-Bousfield* if it is *$H_*(-;R)$-local* in the sense of [1, §1], that is, if the natural map $X \to X_R$ is a homotopy equivalence. Similarly, a group π is called R-Bousfield if it is HR-local in the sense of [1, 5.1], and a π-module M is R-Bousfield if it is HR-local as an (abelian) group and $H\mathbf{Z}$-local [1, 5.3] as a π-module.

In these terms, theorem 5.5 of [1] reads that a connected space X is R-Bousfield iff $\pi_1 X$ is R-Bousfield and the higher homotopy groups of X are R-Bousfield as $\pi_1 X$-modules.

2. A construction of X_R by successive approximation

The idea of the construction will be to start with the simplest possible map (the map from X to a one-point space) and iteratively modify the range of this map until an R-homology isomorphism is obtained.

2.1. *The Dold–Kan construction.* For any space X, $R \otimes X$ will denote the mod R *Dold–Kan constructon* on X, that is, $R \otimes X$ is the simplicial R-module which, for each $n \geqq 0$, has as its set of n-simplices the free R-module on the n-simplices of X [3, p. 14]. If (Y, X) is a simplicial pair then $R \otimes (Y, X)$ will denote the quotient simplicial R-module pair $(R \otimes Y / R \otimes X, 0)$.

The homotopy groups of $R \otimes X$ are naturally isomorphic to the mod R homology groups of X, and the natural inclusion $X \to R \otimes X$ induces a map on homotopy which is essentially the Hurewicz homomorphism. There are similar relative statements.

Suppose that $f: X \to Y$ is an arbitrary map of spaces. The *mapping cylinder* of f, denoted $\mathrm{Cyl}(f)$, is defined as the pushout of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ X \times \Delta[1] & \longrightarrow & \mathrm{Cyl}(f) \end{array}$$

where $\Delta[1]$ is the standard 1-simplex [3, p. 234] and i_1 is the inclusion $x \mapsto (x, \langle 1 \rangle)$. The alternate inclusion $i_0 : X \to X \times \Delta[1]$ given by $x \mapsto (x, \langle 0 \rangle)$ induces an isomorphism of X onto a subcomplex X_0 of $\mathrm{Cyl}(f)$. The collapsed pair $(\mathrm{Cyl}(f)/X_0,\ X_0/X_0)$ is by definition $\mathrm{Cone}(f)$, the *mapping cone* of f. Since $R\otimes\mathrm{Cone}(f)$ is isomorphic to the quotient simplicial R-module pair $(R\otimes\mathrm{Cyl}(f)/R\otimes X_0, 0)$, the homotopy groups of $R\otimes\mathrm{Cone}(f)$ fit into the long exact sequence

$$\cdots\to H_i(X;R)\xrightarrow{f_*}H_i(Y;R)\longrightarrow\pi_i(R\otimes\mathrm{Cone}(f))\xrightarrow{\partial}H_{i-1}(X;R)\to\cdots$$

2.2. LEMMA. *For any map f, $R\otimes\mathrm{Cone}(f)$ is both a Kan complex and an R-Bousfield space. Moreover, the natural map* $\mathrm{Cone}(f)\to R\otimes\mathrm{Cone}(f)$ *induces an injection on* mod R *homology groups.*

PROOF. Since it is a simplicial R-module, $R\otimes\mathrm{Cone}(f)$ is a Kan complex [6, p. 67] which is homotopy equivalent to a product of R-module Eilenberg–MacLane spaces [6, p. 106]. By [1, 5.5], $R\otimes\mathrm{Cone}(f)$ is R-Bousfield. The last statement of the lemma follows as in [6, p. 97] from the fact that the induced map $R\otimes\mathrm{Cone}(f)\to R\otimes(R\otimes\mathrm{Cone}(f))$ has a left inverse given by evaluating formal sums.

2.3. *R-modification.* The *path space* $\Lambda(Y,*)$ of a pointed space $(Y,*)$ is defined to be the standard function complex of maps of pairs [6, p. 17]

$$\Lambda(Y,*)=\mathit{Hom}\,((\Delta[1],\langle 0\rangle),\quad(Y,*)).$$

As usual, the inclusion $\langle 1\rangle\to\Delta[1]$ induces a projection $\Lambda(Y,*)\to Y$, which is a Kan fibration if Y is a Kan complex.

Given $f: X\to Y$, let Y' be the pullback of the square

$$\begin{array}{ccc} Y' & \longrightarrow & \Lambda(R\otimes\mathrm{Cone}(f)) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & R\otimes\mathrm{Cone}(f) \end{array}$$

where the right vertical map is path space projection and the bottom map is the composite $Y\to\mathrm{Cone}(f)\to R\otimes\mathrm{Cone}(f)$. Note that the composite $X\times\Delta[1]\to\mathrm{Cone}(f)\to R\otimes\mathrm{Cone}(f)$ provides a map $X\to\Lambda(R\otimes\mathrm{Cone}(f))$ which combines with the original map f to give a map $f': X\to Y'$. This map f' is called the *R-modification* of f; it fits into a commutative diagram

$$\begin{array}{ccc} & \overset{f'}{\nearrow} & Y' \\ X & & \downarrow \\ & \underset{f}{\searrow} & Y \end{array}$$

The space Y' is just obtained by repairing Y by the extent to which its mod R homology differs from that of X. In this process new homological discrepancies are usually introduced, but there is a sense in which they are independent of the old ones (cf. [2, 3.1, 3.2]).

2.4. Lemma. *If f' is the R-modification of f, the natural map $H_*(\mathrm{Cone}(f'); R) \to H_*(\mathrm{Cone}(f); R)$ is zero.*

Proof. Given $f: X \to Y$, let Z be the pullback of the obvious square

$$\begin{array}{ccc} Z & \longrightarrow & \Lambda(R \otimes \mathrm{Cone}(f)) \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(f) & \longrightarrow & R \otimes \mathrm{Cone}(f) \end{array}$$

and let $g: X \to Z$ be the map which is determined by $i_0: X \to \mathrm{Cyl}(f)$ and the trivial map $X \to \Lambda(R \otimes \mathrm{Cone}(f))$. The maps g and i_0 are both inclusions; moreover, up to weak homotopy type Z is the same as $Y' = \mathrm{range}(f')$ and $\mathrm{Cyl}(f)$ is the same as Y. An easy diagram argument reduces the lemma to showing that the relative homology map

$$\begin{aligned} H_*(Z, g(X); R) &\to H_*(\mathrm{Cyl}(f), i_0(X); R) \\ &= \tilde{H}_*(\mathrm{Cone}(f); R) \end{aligned}$$

is trivial. This follows from the second part of 2.2 and the fact that the evident map $Z/g(X) \to R \otimes \mathrm{Cone}(f)$ is explicitly null homotopic, since it lifts to a map $Z/g(X) \to \Lambda(R \otimes \mathrm{Cone}(f))$.

2.5. *The long tower.* Let Ω be the opposite category of the category of all ordinals, that is, Ω has one object for each ordinal β and one morphism $\alpha \to \beta$ for each $\alpha \geqq \beta$. A *long tower* in a category $\mathbf{C}$ is a functor $F: \Omega \to \mathbf{C}$, usually written $\{F(\alpha)\}_\alpha$. A *map* $f: \{X_\alpha\}_\alpha \to \{Y_\alpha\}_\alpha$ of long towers is a natural transformation of functors, with components $f_\alpha: X_\alpha \to Y_\alpha$. The long tower $\{X_\alpha\}_\alpha$ is said to be *augmented* by the object X of $\mathbf{C}$ if there is a map from the constant long tower $\{X\}_\alpha$ into $\{X_\alpha\}_\alpha$.

Similarly, a *tower* $\{F(\alpha)\}_{\alpha<\beta}$ of length β in $\mathbf{C}$ is a functor $F: \Omega_\beta \to \mathbf{C}$, where Ω_β is the full subcategory of Ω consisting of all ordinals less than β. Unlike a long tower, a tower is a small (= set-indexed) diagram; this means that a tower of spaces, for instance, has an inverse limit.

For any space X, define a functorial long tower $\{X^R_\alpha\}_\alpha$ of spaces, naturally augmented by $f: X \to \{X^R_\alpha\}$, in the following inductive say:

(2.6) (i) $f_0 : X \to X_0^R$ is the unique map of X to a one-point space,

(ii) if $\alpha = \beta + 1$ is a successor ordinal, then $f_\alpha : X \to X_\alpha^R$ is the R-modification of f_β,

(iii) if α is a limit ordinal, then $f_\alpha : X \to X_\alpha^R$ is the natural map of X to $\varprojlim \{X_\beta^R\}_{\beta<\alpha}$.

2.7. THEOREM. *The long tower* $\{X_\alpha^R\}_\alpha$ *is a long R-homology localization tower for X in the sense that it has the following properties*:

(i) *the spaces* X_α^R, $\alpha \in \Omega$ *are R-Bousfield,*

(ii) *if* $f : X \to Y$ *induces an isomorphism* $H_*(X;R) \to H_*(Y;R)$, *then f induces homotopy equivalences* $X_\alpha^R \to Y_\alpha^R$, $\alpha \in \Omega$,

(iii) *there is some ordinal* β, *depending on X, such that for all* $\alpha \geqq \beta$ *the map* $X \to X_\alpha^R$ *is, up to homotopy, the Bousfield* $H_*(-;R)$*-localization map* $X \to X_R$.

2.8. REMARK. The inductive construction of (2.6) can be carried out with any map $g : X \to Y$ replacing the initial map of X to a point. It seems safe to conjecture that up to homotopy this construction always gives, for large enough ordinals, the general Bousfield factorization of g into the composite of a mod R *homology equivalence* and an $H_*(\ ;R)$*-fibration* [1, 11.1].

2.9. REMARK. By construction, the long tower $\{X_\alpha^R\}_\alpha$ has the property that for each ordinal α the map $X_{\alpha+1}^R \to X_\alpha^R$ is a principle fibration with a simplicial R-module as fibre. In fact, $X_{\alpha+1}^R$ is the pullback of a fibre square

$$\begin{array}{ccc} X_{\alpha+1}^R & \longrightarrow & \Lambda(R \otimes \mathrm{Cone}(f_\alpha)) \\ \downarrow & & \downarrow \\ X_\alpha^R & \longrightarrow & R \otimes \mathrm{Cone}(f_\alpha) \end{array}$$

in which the right-hand vertical map is a map of simplicial R-modules. Alternatively, if α is a limit ordinal it follows from [4, 4.2] that X_α^R has the homotopy type of the homotopy inverse limit [3, p. 295] of the tower $\{X_\beta^R\}_{\beta<\alpha}$. In this way 2.7 gives a short proof of the following result, which was proved (with minor changes) by *ad hoc* arguments in [4, §5].

2.10. COROLLARY. *Up to homotopy, the class of R-Bousfield spaces is the smallest class of spaces which contains all simplicial R-modules and is closed under arbitrary homotopy inverse limits.*

The fact that the class of R-Bousfield spaces contains the class described in 2.10 follows from [1: §5, §12].

Proof of 2.7. Parts (i) and (ii) of 2.7 are proved by induction on the ordinal α. If α is a successor ordinal the statements follow from the fact that the class of R-Bousfield spaces is closed under fibration pullbacks [1, 12.7] and the observation that if

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$

is a commutative diagram in which the vertical maps are R-homology equivalences, then the induced map $R \otimes \mathrm{Cone}(f) \to R \otimes \mathrm{Cone}(g)$ is a homotopy equivalence. If α is a limit ordinal, the towers $\{X^R_\beta\}_{\beta<\alpha}$ and $\{Y^R_\beta\}_{\beta<\alpha}$ are clearly *fibrant* in the sense of [4, §4], so that X^R_α and Y^R_α have the homotopy type of $\underleftarrow{\mathrm{holim}}\{X^R_\beta\}_{\beta<\alpha}$ and $\underleftarrow{\mathrm{holim}}\{Y^R_\beta\}_{\beta<\alpha}$. The desired statements then follow from the homotopy invariance property of homotopy inverse limits [3, p. 28] and the fact that the class of R-Bousfield spaces is closed under homotopy inverse limits [1, 12.9].

The proof of 2.7 (iii) will be given in §5.

3. Long towers of groups and modules

The purpose of this section is to show that some algebraic results which are well-known for countable towers of R-nilpotent groups and R-nilpotent π-modules [3, III] also hold for long towers of R-Bousfield groups and R-Bousfield π-modules. The principle behind this is that an R-Bousfield group or π-module behaves in some sense alike an R-nilpotent group or π-module with a very long (i.e. possibly transfinite) R-lower central series filtration [2, §3, §8].

3.1. *Preliminaries.* Let $f:\{A_\alpha\}_\alpha \to \{B_\alpha\}_\alpha$ be a map between two long towers of objects in a category C. The map f is said to be a *pro-isomorphism* if for each ordinal β there is an $\alpha > \beta$ and a map $B_\alpha \to A_\beta$ such that the diagram

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f_\alpha} & B_\alpha \\ \downarrow & \swarrow & \downarrow \\ A_\beta & \xrightarrow{f_\beta} & B_\beta \end{array}$$

commutes. If C is a pointed category (that is, a category with a distinguished object $*$ which is both initial and terminal), then the long tower $\{A_\alpha\}_\alpha$ is said to be *pro-trivial* if the unique map $\{*\}_\alpha \to \{A_\alpha\}_\alpha$ of the trivial constant long tower into $\{A_\alpha\}_\alpha$ is a pro-isomorphism. This is equivalent to the condition that the unique map $\{A_\alpha\}_\alpha \to \{*\}_\alpha$ be a pro-isomorphism.

If C is a category of groups or of modules over some ring, then a map $f:\{A_\alpha\}_\alpha \to \{B_\alpha\}_\alpha$ of long towers in C is said to be a *pro-monomorphism* or a *pro-epimorphism* if the long towers $\{\ker f_\alpha\}_\alpha$ or $\{\operatorname{coker} f_\alpha\}_\alpha$, respectively, are pro-trivial. (If C is a category of groups, then $\{\operatorname{coker} f_\alpha\}_\alpha$ is a long tower of pointed sets.) It is easy to check that in this case f is a pro-isomorphism iff it is both a pro-monomorphism and a pro-epimorphism. In fact, all of the elementary algebraic properties of towers indexed by the positive integers, including the evident analogue of the five lemma [3, p. 75], also hold for long towers in C.

3.2. THEOREM. *Let $f:\{\pi_\alpha\}_\alpha \to \{\sigma_\alpha\}_\alpha$ be a map of long towers of R-Bousfield groups. Then f is a pro-isomorphism if the induced map $\{H_i(\pi_\alpha; R)\}_\alpha \to \{H_i(\sigma_\alpha; R)\}_\alpha$ is a pro-isomorphism for $i = 1$ and a pro-epimorphism for $i = 2$.*

If $\{\pi_\alpha\}_\alpha$ is a long tower of groups, then a long tower $\{A_\alpha\}_\alpha$ of *modules over* $\{\pi_\alpha\}_\alpha$ is by definition a long tower of abelian groups such that

(i) each A_α is a module over π_α, and

(ii) the diagrams

$$\begin{array}{ccc} \pi_\alpha \times A_\alpha & \longrightarrow & A_\alpha \\ \downarrow & & \downarrow \\ \pi_\beta \times A_\beta & \longrightarrow & A_\beta \end{array} \qquad \alpha \geqq \beta$$

commute (where the horizontal maps are action maps and the vertical maps are induced by tower maps).

A *map* $f:\{A_\alpha\}_\alpha \to \{B_\alpha\}_\alpha$ of long towers of modules over $\{\pi_\alpha\}_\alpha$ is a map of long towers of abelian groups such that for each ordinal α the component map $f_\alpha : A_\alpha \to B_\alpha$ is a map of π_α-modules.

3.3. THEOREM. *Let $\{\pi_\alpha\}_\alpha$ be a long tower of groups, and let $f:\{A_\alpha\}_\alpha \to \{B_\alpha\}_\alpha$ be a map of long towers of modules over $\{\pi_\alpha\}_\alpha$. Suppose that, for each ordinal α, A_α and B_α are R-Bousfield π_α-modules. Then*

(i) the induced map $\{A_\alpha \otimes R\}_\alpha \to \{B_\alpha \otimes R\}_\alpha$ is a pro-isomorphism if the induced map $\{H_i(\pi_\alpha; A_\alpha \otimes R)\}_\alpha \to \{H_i(\pi_\alpha; B_\alpha \otimes R)\}_\alpha$ is a pro-isomorphism for $i = 0$ and a pro-epimorphism for $i = 1$, and

(ii) f itself is a pro-isomorphism if the induced map $\{A_\alpha \otimes R\}_\alpha \to \{B_\alpha \otimes R\}_\alpha$ is

a pro-isomorphism and the induced map $\{H_0(\pi_\alpha\,;A_\alpha * R)\}_\alpha \to \{H_0(\pi_\alpha\,;B_\alpha * R)\}_\alpha$ *is a pro-epimorphism.*

3.4. Remark. The symbols $\otimes$ and $*$ denote tensor or torsion product over the ring $\mathbf{Z}$ of integers. If π acts on A, then π acts on $A\otimes R$ and $A * R$ via the given action on A and the trivial action on R.

3.5. Remark. It is not hard to extract from 3.3 the statement that f is a pro-isomorphism iff the induced map $\{H_i(\pi_\alpha\,;R\,;A_\alpha)\}_\alpha \to \{H_i(\pi_\alpha\,;R\,;B_\alpha)\}_\alpha$ is a pro-isomorphism for $i=0$ and a pro-epimorphism for $i=1$. (Here $H_i(\pi\,;R\,;A)$ denotes $\mathrm{Tor}_i^{Z[\pi]}(R,A)$.) This is more in line with 3.2 but less convenient for our purposes.

Proof of 3.2. The first step is to prove that f is a pro-epimorphism. To do this it is enough to show that for any ordinal β there is an $\alpha<\beta$ such that image $(\sigma_\alpha\to\sigma_\beta)$ is contained within image $(f_\beta:\pi_\beta\to\sigma_\beta)$.

Let $D_\gamma\sigma_\beta$, $\gamma\in\Omega$, $\gamma\neq 0$ denote the *R-derived series subgroups* of σ_β relative to the map f_β [2, 2.6]. These are defined inductively by

$$D_1\sigma_\beta=\sigma_\beta,$$

$$D_{\gamma+1}\sigma_\beta=\ker\left(D_\gamma\sigma_\beta\to\operatorname{coker}\left(H_1(\pi_\beta\,;R)\to H_1(D_\gamma\sigma_\beta:R)\right)\right),\qquad \gamma\geqq 1$$

$$D_\gamma\sigma_\beta=\bigcap_{\lambda<\gamma}D_\lambda\sigma_\beta,\qquad \gamma\ \text{a limit ordinal.}$$

By [2, 2.11] the image of the map $f_\beta:\pi_\beta\to\sigma_\beta$ is equal to $D_\gamma\sigma_\beta$ for sufficiently large ordinals γ. It is thus sufficient to show that for each γ there is an ordinal $\alpha(\gamma)>\beta$ such that the image of $\sigma_{\alpha(\gamma)}$ in σ_β is contained in $D_\gamma\sigma_\beta$.

This is done by transfinite induction on γ. The case $\gamma=1$ is trivial. If $\gamma=\lambda+1$ then by protriviality of $\{\operatorname{coker}H_1(\pi_\alpha\,;R)\to H_1(\sigma_\alpha\,;R)\}_\alpha$ it is possible to find $\alpha'>\alpha(\lambda)$ such that image$(H_1(\sigma_{\alpha'};R)\to H_1(\sigma_{\alpha(\lambda)};R))$ is contained in image $(H_1(\pi_{\alpha(\lambda)};R)\to H_1(\sigma_{\alpha(\lambda)};R))$. Let $\alpha(\gamma)=\alpha'$. If γ is a limit ordinal, let $\alpha(\gamma)=\sup\{\alpha(\lambda):\lambda<\gamma\}$. It is easy to check that this choice of the ordinals $\alpha(\gamma)$ has the desired properties.

To prove that f is a pro-monomorphism, note that by the argument above it is possible to assume that the maps $f_\alpha:\pi_\alpha\to\sigma_\alpha$ are actually onto, since otherwise $\{\sigma_\alpha\}_\alpha$ could be replaced by the pro-isomorphic tower $\{\text{image}(f_\alpha:\pi_\alpha\to\sigma_\alpha)\}_\alpha$. Let $\kappa_\alpha=\text{kernel}(f_\alpha:\pi_\alpha\to\sigma_\alpha)$. It is enough to show that the long tower $\{\kappa_\alpha\}_\alpha$ is pro-trivial, or in other words that for each ordinal β there is an $\alpha>\beta$ such that the map $\kappa_\alpha\to\kappa_\beta$ is trivial.

Note that the long tower of short exact sequences

$$1 \to \{\kappa_\alpha\}_\alpha \to \{\pi_\alpha\}_\alpha \to \{\sigma_\alpha\}_\alpha \to 1$$

gives rise to a long tower of low dimensional homology exact sequence

$$\{H_2(\pi_\alpha; R)\}_\alpha \to \{H_2(\sigma_\alpha; R)\}_\alpha \to \{H_0(\pi_\alpha; H_1(\kappa_\alpha; R))\}_\alpha \to$$

$$\{H_1(\pi_\alpha; R)\}_\alpha \to \{H_1(\sigma_\alpha; R)\}_\alpha \to 0.$$

Thus the hypotheses imply that the long tower $\{H_0(\pi_\alpha; H_1(\kappa_\alpha; R))\}_\alpha$ is pro-trivial.

Given β, let $\Gamma_\gamma \kappa_\beta$, $\gamma \in \Omega$, $\gamma \neq 0$ denote the *R-lower-central-series subgroups of κ_β relative to the conjugation action of π_β*. These are defined inductively as follows (cf. [2, § 11]):

$$\Gamma_1 \kappa_\beta = \kappa_\beta,$$

$$\Gamma_{\gamma+1}\kappa_\beta = \ker(\Gamma_\gamma \kappa_\beta \to H_0(\pi_\beta; H_1(\Gamma_\gamma \kappa_\beta; R))), \qquad \gamma \geqq 1$$

$$\Gamma_\gamma \kappa_\beta = \bigcap_{\lambda < \gamma} \Gamma_\lambda \kappa_\beta, \qquad \gamma \text{ a limit ordinal.}$$

Since π_β contains no normal subgroup κ such that $H_0(\pi_\beta; H_1(\kappa; R))$ vanishes [2, 1.2], it follows that for all sufficiently large ordinals γ the subgroup $\Gamma_\gamma \kappa_\beta$ of κ_β is trivial. Thus it suffices to show that for each γ there is an $\alpha(\gamma) > \beta$ such that the image of $\kappa_{\alpha(\gamma)}$ in κ_β is contained in $\Gamma_\gamma \kappa_\beta$.

This is done by transfinite induction on γ. The case $\gamma = 1$ is trivial. If $\gamma = \lambda + 1$, then it is possible to find an ordinal $\alpha' > \alpha(\lambda)$ such that the map

$$H_0(\pi_{\alpha'}; H_1(\kappa_{\alpha'}; R)) \to H_0(\pi_{\alpha(\lambda)}; H_1(\kappa_{\alpha(\lambda)}; R))$$

is trivial; let $\alpha(\gamma) = \alpha'$. If γ is a limit ordinal, let $\alpha(\gamma) = \sup(\alpha(\lambda); \lambda < \gamma)$. It is easy to check that this choice of the ordinals $\alpha(\gamma)$ has the desired properties.

3.6. LEMMA. *Let $\{\pi_\alpha\}_\alpha$ be a long tower of groups, and let $\{A_\alpha\}_\alpha$ be a long tower of modules over $\{\pi_\alpha\}_\alpha$. Suppose that for each ordinal α, A_α is an R-Bousfield π_α-module. Then the long tower $\{A_\alpha\}_\alpha$ is pro-trivial if and only if the long tower $\{H_0(\pi_\alpha; A_\alpha \otimes R)\}_\alpha$ is pro-trivial.*

PROOF. If $R \subseteq \mathbf{Q}$, then an abelian group A is HR-local iff it is an R-module, *i.e.* iff $A \otimes R$ is isomorphic to A. Thus in this case a proof of 3.6 can be constructed along the lines of either half of the proof of 3.2, but using instead of [2, 1.2] the fact [2, 6.2] that an $H\mathbf{Z}$-local π-module A contains no submodule κ such that $H_0(\pi; \kappa)$ vanishes.

If $R = \mathbf{Z}/p\mathbf{Z}$, note that if A is an R-Bousfield and therefore $H\mathbf{Z}$-local

π-module, then $A \otimes R = \operatorname{coker}(A \xrightarrow{p} A)$ is also $H\mathbf{Z}$-local [1, 8.6]. Thus the pro-triviality of $\{H_0(\pi_\alpha; A_\alpha \otimes R)\}_\alpha$ together with the argument above shows that $\{A_\alpha \otimes R\}_\alpha = \{H_1(A_\alpha; R)\}_\alpha$ is pro-trivial. Since each A_α is HR-local as an abelian group, the desired result follows from an application of 3.2 to the map $\{A_\alpha\}_\alpha \to \{0\}_\alpha$.

PROOF OF 3.3. The proof consists in repeatedly applying 3.6 to show that the cokernals and kernals of appropriate long tower maps are pro-trivial. The main point to keep in mind is that if $f: A \to B$ is a map of R-Bousfield π-modules, then $\ker f$ and $\operatorname{coker} f$ are also R-Bousfield π-modules. This follows from [2, 1.5, 6.3], except that it must be checked that the cokernal of a map of HR-local abelian groups is HR-local. However, this can easily be proved along the lines of [1, 8.6].

4. The Whitehead theorem

This section contains the proof of the following theorem.

4.1. THEOREM. *Let $f:\{X_\alpha\}_\alpha \to \{Y_\alpha\}_\alpha$ be a map of long towers of pointed R-Bousfield spaces. If f induces pro-isomorphisms*

$$\{H_i(X_\alpha; R)\}_\alpha \to \{H_i(Y_\alpha; R)\}_\alpha \qquad i \geqq 0$$

then f induces pro-isomorphisms

$$\{\pi_i(X_\alpha)\}_\alpha \to \{\pi_i(Y_\alpha)\}_\alpha \qquad i \geqq 0.$$

The following two lemmas are needed to set up an inductive spectral sequence argument.

4.2. LEMMA. *Let $f:\{X_\alpha\}_\alpha \to \{Y_\alpha\}_\alpha$ be a map of long towers of connected pointed spaces, which induces pro-isomorphisms $\{\pi_i X_\alpha\}_\alpha \to \{\pi_i Y_\alpha\}_\alpha$, $i \geqq 0$. Let $\{A_\alpha\}_\alpha$ be a tower of modules over $\{\pi_1 Y_\alpha\}_\alpha$* (see §3). *Then f induces pro-isomorphisms*

$$\{H_i(X_\alpha; A_\alpha)\}_\alpha \to \{H_i(Y_\alpha; A_\alpha\}_\alpha \qquad i \geqq 0.$$

The statement of the lemma is restricted to pointed connected spaces to avoid having to confront what it means to choose a "basepoint" for an arbitrary long tower of spaces. Lemma 4.2 is obvious if $\{X_\alpha\}_\alpha$ and $\{Y_\alpha\}_\alpha$ are long towers of Eilenberg–MacLane spaces of type $K(\pi, n)$, for some fixed n. The general case is treated by forming the induced Postnikov stage maps [6, p. 32]

$$P_nf:\{P_nX_\alpha\}_\alpha\to\{P_nY_\alpha\}_\alpha$$

and proving by induction on n that each P_nf induces pro-isomorphisms on the appropriate homology groups. The induction step depends on looking at the map of long towers of Serre spectral sequences induced by

$$\begin{array}{ccc}\{K(\pi_{n+1}X_\alpha,n+1)\}_\alpha & \to & \{K(\pi_{n+1}Y_\alpha,n+1)\}_\alpha\\ \downarrow & & \downarrow\\ \{P_{n+1}X_\alpha\}_\alpha & \to & \{P_{n+1}Y_\alpha\}_\alpha\\ \downarrow & & \downarrow\\ \{P_nX_\alpha\}_\alpha & \to & \{P_nY_\alpha\}_\alpha\end{array}$$

and repeatedly applying the "five lemma" [3, p. 75] to pass from pro-isomorphisms at E^2 to pro-isomorphisms on the abutment.

4.3. LEMMA. *Let* $\{E^2_{p,q}(X_\alpha)\Rightarrow H_{p+q}(X_\alpha;R)\}_\alpha\xrightarrow{f}\{E^2_{p,q}(Y_\alpha)\Rightarrow H_{p+q}(Y_\alpha;R)\}_\alpha$ *be a map of towers of first quadrant spectral sequences of homological type. If* $H_n(f;R)$ *is a pro-isomorphism for all* n *and* $E^2_{p,q}(f)$ *is a pro-isomorphism for* $q<k$, *then*

(i) $E^2_{0,k}(f)$ *is a pro-isomorphism, and*

(ii) $E^2_{1,k}(f)$ *is a pro-epimorphism.*

This is stated in [3, p. 92] for towers indexed by the natural numbers, but it holds equally well for long towers.

PROOF OF 4.1. We leave it to the reader to verify that the map $\{\pi_0(X_\alpha)\}_\alpha\to\{\pi_0(Y_\alpha)\}_\alpha$ is a pro-isomorphism of pointed sets. By passing to basepoint components it is then possible to assume that the spaces X_α and Y_α are all connected.

The proof proceeds by induction on $n<0$ to show that the induced homotopy map is a pro-isomorphism for $i\leqq n$.

If $n=1$, consider the map of long towers of mod R *homology Serre spectral sequences induced by*

$$\begin{array}{ccc}\{P^1X_\alpha\}_\alpha & \to & \{P^1Y_\alpha\}_\alpha\\ \downarrow & & \downarrow\\ \{X_\alpha\}_\alpha & \to & \{Y_\alpha\}_\alpha\\ \downarrow & & \downarrow\\ \{K(\pi_1X_\alpha,1)\}_\alpha & \to & \{K(\pi_1Y_\alpha,1)\}_\alpha\end{array}$$

(Here P^iZ denotes the i-connective cover of Z [6, p. 33]). Low dimensional exact sequences show that the induced map

$$\{H_i(\pi_1 X_\alpha; R)\}_\alpha \to \{H_i(\pi_1 Y_\alpha; R)\}_\alpha$$

is a pro-isomorphism for $i = 1$ and a pro-epimorphism for $i = 2$. The desired statement then follows from 3.2.

If $n > 1$, consider the map of long towers of mod R homology Serre spectral sequences induced by

$$\begin{array}{ccc} \{P^{n-1}X_\alpha\}_\alpha & \to & \{P^{n-1}Y_\alpha\}_\alpha \\ \downarrow & & \downarrow \\ \{X_\alpha\}_\alpha & \to & \{Y_\alpha\}_\alpha \\ \downarrow & & \downarrow \\ \{P_{n-1}X_\alpha\}_\alpha & \to & \{P_{n-1}Y_\alpha\}_\alpha \end{array}$$

If Z is any connected space and A is a module over $\pi = \pi_1 Z$, then there are natural isomorphisms $H_i(Z; M) \approx H_i(\pi; M)$, $i = 0, 1$. Thus, in connection with 4.2 and the induction hypothesis, 4.3 shows that the natural map

$$\{H_i(\pi_1 X_\alpha; \pi_n X_\alpha \otimes R)\}_\alpha \to \{H_i(\pi_1 Y_\alpha; \pi_n Y_\alpha \otimes R)\}_\alpha$$

is a pro-isomorphism for $i = 0$ and a pro-epimorphism for $i = 1$. It is clear from 4.2 that the pro-isomorphism $\{\pi_1 X_\alpha\}_\alpha \to \{\pi_1 Y_\alpha\}_\alpha$ induces pro-isomorphisms

$$\{H_i(\pi_1 X_\alpha; \pi_n Y_\alpha)\}_\alpha \to \{H_i(\pi_1 Y_\alpha; \pi_n Y_\alpha)\}_\alpha$$

and it follows from [1, 8.8] that each $\pi_n Y_\alpha$ in R-Bousfield as a $\pi_1 X_\alpha$-module. Thus 3.3 implies that the map $\{\pi_n X_\alpha \otimes R\}_\alpha \to \{\pi_n Y_\alpha \otimes R\}_\alpha$ is a pro-isomorphism.

This finishes the inductive step if $R \subseteq \mathbf{Q}$. Otherwise, another application of 4.2 and 4.3 provides (somewhat more than) a pro-epimorphism

$$\{H_0(\pi_1 X_\alpha; H_{n+1}(P^{n-1}X_\alpha; R))\}_\alpha \to \{H_0(\pi_1 Y_\alpha; H_{n+1}(P^{n-1}Y_\alpha; R))\}_\alpha.$$

In view of the universal coefficient epimorphism

$$H_{n+1}(P^{n-1}Z; R) \to \pi_n Z * R \to 0$$

valid for any connected space Z, this gives a pro-epimorphism

$$\{H_0(\pi_1 X_\alpha; \pi_n X_\alpha * R)\}_\alpha \to \{H_0(\pi_1 Y_\alpha; \pi_n Y_\alpha * R)\}_\alpha.$$

Now, by the argument above, the desired inductive step follows from 3.3.

5. Completion of the proof of 2.7

It remains to prove 2.7 (iii). By 2.7 (ii) the $H_*(-;R)$-localization map $X \to X_R$ induces homotopy equivalences $X_\alpha^R \to (X_R)_\alpha^R$ for all ordinals α, so we can assume without loss of generality that X itself is R-Bousfield. It is also convenient to assume that X is connected and pointed; the general case can be handled by successively choosing basepoints in each connected component of X.

It follows from 2.4 that the relative homology long towers

$$\{H_i(\mathrm{Cone}(f_\alpha); R)\}_\alpha \qquad i \geqq 0$$

are all pro-trivial. By a five lemma argument, this implies that the map $f: X \to \{X_\alpha^R\}_\alpha$ induces pro-isomorphisms

$$\{H_i(X; R)\}_\alpha \to \{H_i(X_\alpha^R; R)\}_\alpha \qquad (i \geqq 0).$$

Thus by 4.1, f also induces pro-isomorphisms

$$\{\pi_i X\}_\alpha \to \{\pi_i X_\alpha^R\}_\alpha \qquad (i \geqq 0).$$

Note that in this application of 4.1 the domain $\{X\}_\alpha$ is a constant long tower; the basepoints in the range spaces X_α^R are taken to be the images of the given basepoint of X.

Let ω be the first infinite ordinal. Choose an increasing countable sequence $\alpha(i)$ $(i < \omega)$ of ordinals as follows. Let $\alpha(0) = 0$. Inductively, for $i \geqq 0$ let $\alpha(i+1)$ be some ordinal greater than $\alpha(i)$ such that the dotted arrow

$$\begin{array}{ccc} \pi_j X & \longrightarrow & \pi_j X_{\alpha(i+1)}^R \\ \downarrow & \swarrow & \downarrow \\ \pi_j X & \longrightarrow & \pi_j X_{\alpha(i)}^R \end{array}$$

exists for all $j \geqq 0$. By the definition of pro-isomorphism, such an ordinal exists for any particular j, so that $\alpha(i+1)$ can be chosen as an appropriate least upper bound.

The collection $\{X_{\alpha(i)}^R\}_{i<\omega}$ is a tower of fibrations, and the natural map $X \to \{X_{\alpha(i)}^R\}_{i<\omega}$ induces pro-isomorphisms on all homotopy groups. It follows from [3, p. 251–257] that the map $X \to \varprojlim \{X_{\alpha(i)}^R\}_{i<\omega}$ is a homotopy equivalence. However, the sequence $\alpha(i)$ $(i < \omega)$ of ordinals is *cofinal* [3, p. 317] in the category of all ordinals less than β, where β is $\sup\{\alpha(i) : i < \omega\}$, so that the map

$X^R_\beta = \varprojlim \{X^R_\alpha\}_{\alpha<\beta} \to \varprojlim \{X^R_{\alpha(i)}\}_{i<\omega}$ is actually an isomorphism. Thus the map $X \to X^R_\beta$ is a homotopy equivalence.

There is still the problem of showing that for all ordinals $\gamma \geqq \beta$ the map $X \to X^R_\gamma$ is a homotopy equivalence. This is done by induction on γ. The statement for successor ordinals γ follows easily from the fact that, by definition, the R-modification process leaves unchanged up to homotopy any map which is an R-homology equivalence. For limit γ, a cofinality argument shows that X^R_γ is isomorphic to the inverse limit of a "fibrant" [4, §4] tower $\{X^R_\alpha\}_{\beta \leqq \alpha < \gamma}$ which is constant, up to homotopy. The result then follows from the homotopy invariance property of the homotopy inverse limit [3, p. 287] and the fact that the homotopy inverse limit agrees with the inverse limit for fibrant towers [4, §4].

References

1. A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14**(1975), 135–150.

2. A. K. Bousfield, *Homological localization towers for groups and π-modules*, to appear.

3. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completion and Localizations*, Lecture Notes in Math. No. 304, Springer, New York, 1972.

4. E. Dror and W. G. Dwyer, *A long homology localization tower*, to appear in Comment. Math. Helv.

5. K. Lamotke, *Semisimpliziale algebraische Topologie*, Springer, New York, 1968.

6. J. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, Princeton, 1967.

The Hebrew University of Jerusalem
Jerusalem, Israel

and

Yale University
New Haven, Connecticut, USA