Problem 1.1

2. Soln:

Continuous: If $|x| < \epsilon$, then $|f(x) - 0| = |x||sin(1/x)| \leq \epsilon$, since $|sin(1/x)| \leq 1$. Let $\delta = \epsilon$, $\forall \epsilon > 0$, $\exists \delta = \epsilon$ such that if $|x - 0| < \delta$ then $|f(x) - 0| = |x||sin(1/x)| \leq |x| < \epsilon$. Therefore, $\lim_{x->0} f(x) = 0 = f(0)$.

Not Differentiable: $\lim_{x->0}[f(x) - f(0)]/(x - 0) = \lim_{x->0} sin(1/x)$ does not exist.

16. Soln:

$\ln x = \sum_{k=1}^{1000}(-1)^{k-1}(x-1)^k/k + E_{1000}(x)$ where $E_{1000}(x) = (-1)^{1000}\xi^{-1001}(x-1)^{1001}/1001$ for some ξ with $1 < \xi < x$ and $1 \leq x \leq 2$. For ln2, $x = 2$ and $|E_{1000}(2)| = \xi^{-1001}/1001 \leq 1/1001$ since $\xi > 1$. $\Rightarrow \xi^{-1001} < 1$.

26. Soln: $f(x) = \ln(x + 1)$, $f^{(n)}(x) = (-1)^{n-1}(n - 1)!(1 + x)^{-n}$.

$f(1) = \ln(2)$, $f^{(n)}(1) = (-1)^{n-1}(n - 1)!2^{-n}$.

$\ln(1 + x) = \ln(2) + \sum_{k=1}^{n}(-1)^{k-1}[(x - 1)/2]^k/k + E_n(x)$,

where $E_n(x) = (-1)^n[(x - 1)/(1 + \xi)]^{n+1}/(n + 1)$ for ξ between 1 and x.

$E_n(3) = (-1)^n[2/(1+\xi)]^{n+1}/(n+1)$ for $1 < \xi < 3$. Since $2 < 1+\xi < 4$, $\Rightarrow [2/(1+\xi)]^{n+1} < 1$ and $|E_n(3)| < 1/(n + 1) < 2^{-m}$.

$\Rightarrow n > 2^m - 1$.

35. Soln:

$e = \sum_{k=0}^{n}(1/k!) + E_n(1)$ with $E_n(1) = e^{\xi}/(n + 1)!$, with $0 < \xi < 1$. $\Rightarrow 1 = e^0 < e^{\xi} < e^1 \Rightarrow |E_n| < e/(n + 1)! \leq (6/10) \times 10^{-20}$.

So $n = 21$. Here are 22 terms.

Problem 1.2

6. Soln: Only (d) is true.

8. Soln: Use Taylor expansion.

$e^h = 1 + h + h^2/2! + h^3/3! + ... = 1 + \mathcal{O}(h) = 1 + o(h)$.

$(1 - h^4)^{-1} = 1 + h^4 + h^8 + ... = 1 + \mathcal{O}(h^4) = 1 + o(h^3)$.

$\cos(h) = 1 - h^2/2! + h^4/4! + ... = 1 + \mathcal{O}(h^2) = 1 + o(h)$.

$1 + \sin(h^3) = 1 + h^3 - h^9/3! + ... = 1 + \mathcal{O}(h^3) = 1 + o(h^2)$.

28. Soln:

If $x_n = x + o(1)$, then $x_n - x = o(1)$. $\Rightarrow \lim_{n->\infty}(x_n - x)/1 = 0$. Or $\lim_{n->\infty} x_n = x$.

If $\lim_{n->\infty} x_n = x$, then $\lim_{n->\infty}(x_n - x)/1 = 0$. $\Rightarrow x_n - x = o(1)$ from definition, or $x_n = x + o(1)$.

30. Soln: $\sum_{k=0}^{n} x^k = \sum_{k=0}^{\infty} x^k - \sum_{k=n+1}^{\infty} x^k = 1/(1-x) - x^{n+1}/(1-x)$.

So $\sum_{k=0}^{n} x^k - 1/(1-x) = -x^{n+1}/(1-x) = o(x^n)$ as $x \to 0$,

since $\lim_{x\to 0} x^{n+1}/[(x-1)x^n] = \lim_{x\to 0} x/(x-1) = 0$.