HW7 Solution

1. Problem 4.4.3

Soln: $||x||_1 = \sum_{i=1}^n |x_i| \leq \sum_{i=1}^n max_{1\leq j \leq n}|x_j| \leq \sum_{i=1}^n ||x||_\infty = n||x||_\infty$.

2. Problem 4.4.11

Soln: Let $||x|| = \sum_{i=1}^n |x_i|$. Then $||A|| = \sup\{||Ax|| : ||x|| = 1\}$. Let A_j denote the j^{th} column vector of A. $||Ax|| = ||\sum x_j A_j|| \leq \sum ||x_j A_j|| \leq \sum |x_j| ||A_j|| \leq (\max_j ||A_j||) \sum_{j=1}^n |x_j| \leq \max_j ||A_j||$. Hence $||A|| \leq \max_j ||A_j||$. Now select k so that $||A_k|| = \max_j ||A_j||$. Let $x = (0, ..., 0, 1, 0, ...0)$ with 1 in position k. $||x|| = 1$. Hence $||A|| \geq ||Ax|| = ||\sum x_j A_j|| = ||A_k|| = \max_j ||A_j||$.

3. Problem 4.4.12

Soln: Small $\epsilon > 0$. $||A||_1 = 2 + \epsilon$. $A^{-1} = \epsilon^2 \begin{pmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{pmatrix}$. $||A^{-1}||_1 = (2+\epsilon)/\epsilon^2$.

$k(A) = ||A||_1 ||A^{-1}||_1 = (2/\epsilon + 1)^2$.

4. Problem 4.4.41

Soln: (a) $||A||_2 = \sup_{||x||_2=1} ||Ax||_2 \leq \sup \sqrt{n}||Ax||_\infty \leq \sup_{||x||_2=1} \sqrt{n}||A||_\infty \leq \sqrt{n}||A||_\infty$. Hence $n^{-1}||A||_2 \leq n^{-1/2}||A||_\infty$.

(b) $||A||_\infty = \sup ||Ax||_\infty/||x||_\infty \leq \sup ||Ax||_2/||x||_\infty \leq \sup \sqrt{n}||Ax||_2/||x||_2 \leq \sup \sqrt{n}||A||_2 \leq \sqrt{n}||A||_2$. Hence $n^{-1/2}||A||_\infty \leq ||A||_2$.

(c) $||A||_2 = \sup ||Ax||_2/||x||_2 \leq \sup \sqrt{n}||Ax||_1/||x||_2 \leq \sup \sqrt{n}||Ax||_1/||x||_1 \leq \sup \sqrt{n}||A||_1 \leq \sqrt{n}||A||_1$. So $||A||_2 \leq \sqrt{n}||A||_1$.

(d) $||x||_1^2 = (\sum |x_i|)^2 \leq (\sum |x_i|^2)(\sum 1^2) \leq ||x||_2^2 n$. Then $||x||_1 \leq \sqrt{n}||x||_2$. $||A||_1 = \sup_{||x||_1=1} ||Ax||_1 \leq \sup \sqrt{n}||Ax||_2 \leq \sup \sqrt{n}||Ax||_2 = \sqrt{n}||A||_2$. So $n^{1/2}||A||_1 \leq n||A||_2$.

5. 4.4.42

Soln: $k(A) = ||A||_\infty ||A^{-1}||_\infty = 1207.01$. $||x - \hat{x}||/||x|| \leq k(A)(||b - \hat{b}||/||b||)$.

$b = (4, 4)^T, x = (404, 0)^T$. $\hat{b} = (3, 5)^T, \hat{x} = (397, 200)^T$. $||x - \hat{x}||_\infty = 200$, $||x||_\infty = 404$. $||x - \hat{x}||/||x|| = 0.49$. $(||b - \hat{b}||/||b||) = 0.25$.

6. Problem 4.6.2

Soln: $G = I - A$. Let λ be any eigenvalue of G. Let x be a corresponding eigenvector, $||x||_\infty = 1$. $Gx = \lambda x \Rightarrow -\sum_{j\neq i, j=1}^n a_{ij}x_j + (1 - a_{ii})x_i = \lambda x_i, 1 \leq i \leq n$. Select i such that $|x_i| = 1 \geq |x_j|$ for all j, then $|\lambda| \leq \sum_{j\neq i, j=1}^n |a_{ij}| + |1 - a_{ii}|$. Since A is row diagonally dominant, $|\lambda| < 1$.

7. The iteration formula is $x^{(k+1)} = Gx^{(k)} + d$. If $||G|| < 1$, show $||x^{(k)} - x|| \leq \frac{||G||}{1-||G||}||x^{(k)} - x^{(k-1)}||$, where x is the accurate solution.

Soln: Assume $m > k$, $\Rightarrow x^{(k)} - x^{(m)} = \sum_{i=k}^{m-1}(x^{(i)} - x^{(i+1)})$,

$||x^{(k)} - x^{(m)}|| \leq \sum_{i=k}^{m-k}||x^{(k+i-1)} - x^{(k+i)}||$.

$||x^{(k)} - x^{(m)}|| \leq \sum_{i=k}^{m-k}||G||^i||x^{(k-1)} - x^{(k)}|| = ||G||\frac{1-||G||^{m-k}}{1-||G||}||x^{(k-1)} - x^{(k)}||$.

Let $m \to \infty$. $||x^{(k)} - x^{(m)}|| \leq \frac{||G||}{1-||G||}||x^{(k)} - x^{(k-1)}||$.