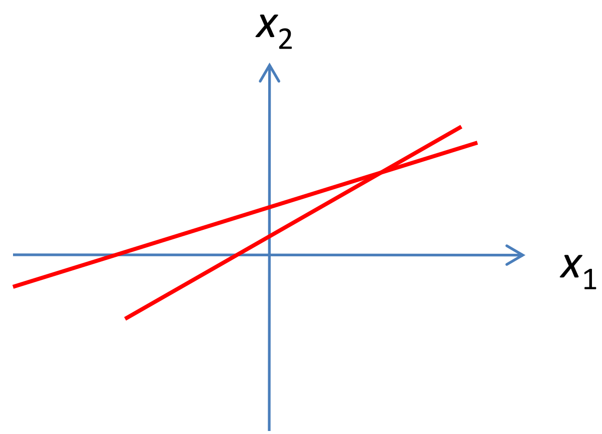
What's a system of linear equations (Section 1.1)

An example from geometry: intersection of two lines



$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

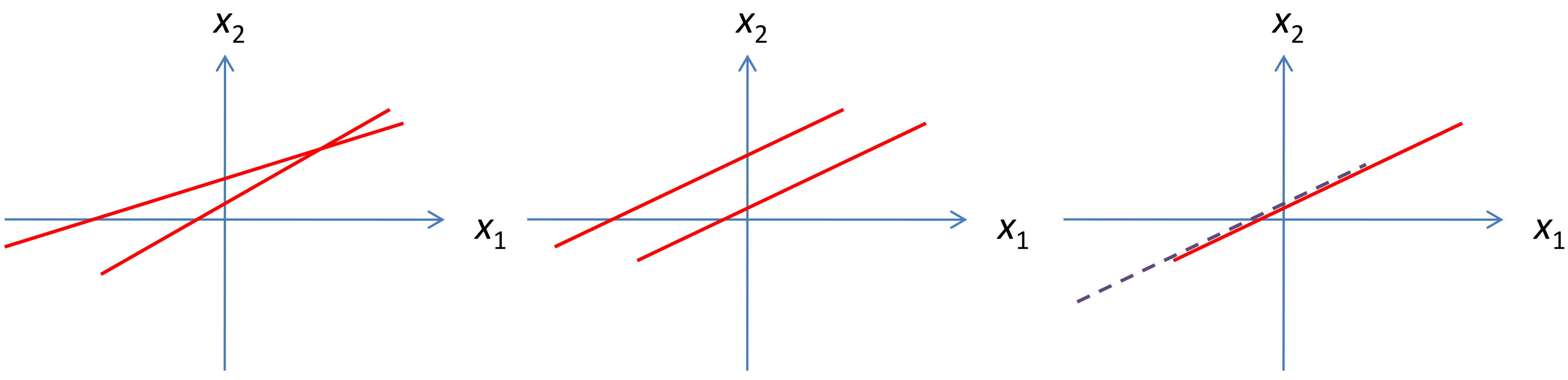
Nonlinear equations: $x_1 - 2x_2 = x_1 x_2$

or $x_1 = \sqrt{x_1} + x_2$ or $x_1 = \sin(x_2) + x_2$

or …

Solution to a system of linear equations

An example from geometry: intersection of two lines



(a) (b) (c)

(a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

Consistent: one solution or infinitely many solutions.

In consistent: no solution.

Existence and Uniqueness questions: 1. Does at least one solution exist **(consistency)**?

2. If solution exists, is it the only one **(uniqueness)**?

Moving toward matrix notation

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

Coefficient matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Size of matrix: 3×3 3×4

Solving by method of elimination

$$x_1 - 2x_2 + x_3 = 0 \qquad (1)$$

$$2x_2 - 8x_3 = 8 \qquad (2)$$

$$-4x_1 + 5x_2 + 9x_3 = -9 \qquad (3)$$

Solution: See class notes

Echelon form of matrix (Section 1.2)

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

Leading entries

$$\begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & 1 & -3 & 8 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form:

1. All nonzero rows are above rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & -8 & 8 \\ 0 & 1 & 0 & -9 \end{bmatrix}$$

Not in echelon form

Reduced echelon form of matrix

$$\begin{bmatrix} 1 & 0 & 0 & 25/2 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot positions

Pivot columns

Reduced echelon form: in addition to be in the echelon form

1. The leading entry in each nonzero row is 1.
2. Each leading entry is the only nonzero entry in the column.

Theorem 1: Uniqueness of the reduced echelon form
Each matrix is row equivalent to one and only one reduced echelon matrix.

Row reduction algorithm to obtain echelon form and reduced echelon form: see class notes.

Existence and uniqueness from echelon form

Ex. Determine the existence and uniqueness of the solution to the system

$$3x_2 - 6x_3 + 6x_4 + 4x_5 = -5$$
$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9$$
$$3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15$$

Solution: The echelon form of the augmented matrix is

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The basic variables are x_1, x_2, and x_5; free variables are x_3 and x_4.
No equation like 0 = const. exits. ⟶ Solution exists (consistent).
Solution has free variables. ⟶ not unique.

Thm 2. Existence and uniqueness

1. A linear system is consistent if and only if echelon form of the augmented matrix has no row like [0, 0, ….0, b]. Here $b \neq 0$
2. A linear system is consistent. Then either (i) it has unique solution when there is no free variables; or (ii) it has infinitely many solutions when there is(are) free variable(s).