Theorems in Sections 5.1-5.2

Theorem 1. Eigenvalues of triangular matrix are entries on its main diagonal.

Example.

$$\boldsymbol{A}=\begin{bmatrix}3 & 6 & -8\\ 0 & 0 & 6\\ 0 & 0 & 2\end{bmatrix} \qquad \boldsymbol{B}=\begin{bmatrix}4 & 0 & 0\\ -2 & 1 & 0\\ 5 & 3 & 4\end{bmatrix}$$

Solution: Eigenvalues of ***A*** are 3, 0, 2. Eigenvalues of ***B*** are 4 and 1.

Question: What does it mean for ***A*** to have an eigenvalue of 0?

$$\boldsymbol{A}\vec{\boldsymbol{x}} = 0\vec{\boldsymbol{x}} \Rightarrow \boldsymbol{A}\vec{\boldsymbol{x}} = \vec{0}\,. \quad \vec{\boldsymbol{x}}$$ is a nonzero vector.

Fact: 0 is an eigenvalue of ***A*** if and only if ***A*** is **not invertible.**

Theorem 2. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors that correspond to ***distinct*** eigenvalues $\lambda_1, \ldots, \lambda_p$ of a n × n matrix ***A***, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

Example.

$$\boldsymbol{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad \lambda_1 = 1, \boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \lambda_2 = 2, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}; \lambda_3 = 3, \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix};$$

Verify the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Similarity. Let ***A*** and ***B*** be n × n matrices. ***A*** and ***B*** are **similar** if there is an **invertible** matrix P s.t. $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B}$ (or equivalently $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$)

Example.

$$\boldsymbol{A} = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} \qquad \boldsymbol{P} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 2 & -1 & -1/2 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Verify $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B}$.

Theorem 4. If n × n matrices ***A*** and ***B*** are similar, then they have the same characteristic polynomials and hence the same eigenvalues (with the same mulitplicities).

Example.

$$\boldsymbol{A} = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Show the characteristic polynomial of ***A*** $= (\lambda + 1)^2(\lambda - 3)$.