Orthogonal sets(Section 6.2)

A set of vectors {$\mathbf{u}_1$, …, $\mathbf{u}_p$} in R^n is an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, i.e., if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever i ≠ j.

Example 1. Show that S = {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$} is an orthogonal set, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}; \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}; \boldsymbol{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Solution: $\mathbf{u}_1 \cdot \mathbf{u}_2 = 3(-1) + 1(2) + 1(1) = 0$

$\mathbf{u}_2 \cdot \mathbf{u}_3 = (-1)(-1/2) + 2(-2) + 1(7/2) = 0$

$\mathbf{u}_1 \cdot \mathbf{u}_3 = 3(-1/2) + 1(-2) + 1(7/2) = 0$

Since each pair of distinct vectors is orthogonal, {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$} is an orthogonal set.

Theorem 4. If S = {$\mathbf{u}_1$, …, $\mathbf{u}_p$} in R^n is an orthogonal set of nonzero vectors, then S is **linearly independent** and **hence is a basis for the subspace spanned** by S.

Definition: An **orthogonal basis** for a subspace W of R^n is a basis for W that is also an orthogonal set.

Theorem 5. Let {$\mathbf{u}_1$, …, $\mathbf{u}_p$} be an **orthogonal basis** for a subspace W of R^n, For each **y** in W, the weights in the linear combination $\boldsymbol{y} = \boldsymbol{c}_1\boldsymbol{u}_1 + \ldots + \boldsymbol{c}_p\boldsymbol{u}_p$ are given by

$$\boldsymbol{c}_j = \frac{\boldsymbol{y} \bullet \boldsymbol{u}_j}{\boldsymbol{u}_j \bullet \boldsymbol{u}_j} \qquad \boldsymbol{j} = 1, \ldots, \boldsymbol{p}$$

Orthonormal sets and matrices with orthonormal columns

Orthonormal sets: an orthogonal set of unit vectors

If W is a subspace spanned by a orthonormal set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal basis** for W.

Example 5: 1. The standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a orthonormal basis for R^n.

2. Show that S = $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of R^3, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}; \boldsymbol{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}; \boldsymbol{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -2/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}.$$

Remark: When vectors in an orthogonal set of nonzero vectors are normalized to have unit length, new vectors are still orthogonal. (Compare Examples 1 and 5.)

Example 6. Show $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$, $||\mathbf{Ux}|| = ||\mathbf{x}||$, where $\boldsymbol{U} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ $\quad \boldsymbol{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$

Theorem 6. An $m \times n$ matrix **U** has orthonormal columns if and only if $\mathbf{U}^T\mathbf{U} = \boldsymbol{I}$

Theorem 7. Let **U** be $m \times n$ matrix with orthonormal columns, let **x** and **y** be in R^n, then

(a) $||\mathbf{Ux}|| = ||\mathbf{x}||$,

(b) $(\mathbf{Ux}) \cdot (\mathbf{Uy}) = \mathbf{x} \cdot \mathbf{y}$,

(c) $(\mathbf{Ux}) \cdot (\mathbf{Uy}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Remark: (a) and (c) say that linear mapping **x** -> **Ux** preserves length and orthogonality.