Orthogonal Decomposition (Section 6.3)

Theorem 8 (Orthogonal decomposition Theorem). Let \( W \) be a subspace of \( \mathbb{R}^n \). Then each vector \( y \) in \( \mathbb{R}^n \) can be written uniquely in the form of

\[
y = \hat{y} + z
\]

where \( \hat{y} \) is in \( W \) and \( z \) is orthogonal to \( W \). If \( \{u_1, \ldots, u_p\} \) is an orthogonal basis of \( W \), then

\[
\text{proj}_W y = \hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \ldots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p
\]

\[
z = y - \hat{y}
\]

Properties of orthogonal projections
1. If \( y \) is in \( W = \text{Span}\{u_1, \ldots, u_p\} \) with \( \{u_1, \ldots, u_p\} \) an orthogonal basis of \( W \), then \( \text{proj}_W y = y \)

2. The best approximation theorem. Let \( W \) be a subspace of \( \mathbb{R}^n \), and \( \hat{y} \) the orthogonal projection of \( y \) onto \( W \). Then \( \hat{y} \) is the closest point in \( W \) to \( y \), in the sense that

\[
\| y - \hat{y} \| < \| y - v \|
\]

for all vectors \( v \) in \( W \) distinct from \( \hat{y} \).