Orthogonal Decomposition (Section 6.3)

Theorem 8 (Orthogonal decomposition Theorem). Let W be a subspace of R^n . Then each vector **y** in R^n can be written **uniquely** in the form of

$$\boldsymbol{y} = \hat{\boldsymbol{y}} + \boldsymbol{z}$$

where $\hat{\boldsymbol{y}}$ is in W and $\boldsymbol{z}$ is orthogonal to W. If {$\mathbf{u}_1$, …, $\mathbf{u}_p$} is **an orthogonal basis** of W, then

$$\boldsymbol{proj}_W \boldsymbol{y} = \hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \bullet \boldsymbol{u}_1}{\boldsymbol{u}_1 \bullet \boldsymbol{u}_1}\boldsymbol{u}_1 + \frac{\boldsymbol{y} \bullet \boldsymbol{u}_2}{\boldsymbol{u}_2 \bullet \boldsymbol{u}_2}\boldsymbol{u}_2 + \ldots + \frac{\boldsymbol{y} \bullet \boldsymbol{u}_p}{\boldsymbol{u}_p \bullet \boldsymbol{u}_p}\boldsymbol{u}_p$$

$$\boldsymbol{z} = \boldsymbol{y} - \hat{\boldsymbol{y}}$$

Properties of orthogonal projections

1. If **y** is in W = Span{$\mathbf{u}_1$, …, $\mathbf{u}_p$} with {$\mathbf{u}_1$, …, $\mathbf{u}_p$} an orthogonal basis of W, then $\boldsymbol{proj}_W \boldsymbol{y} = \boldsymbol{y}$

2. **The best approximation theorem.** Let W be a subspace of R^n, and $\hat{\boldsymbol{y}}$ the orthogonal projection of **y** onto W. Then $\hat{\boldsymbol{y}}$ is the closest point in W to **y,** in the sense that

$$\| \boldsymbol{y} - \hat{\boldsymbol{y}} \| < \| \boldsymbol{y} - \boldsymbol{v} \|$$

for all vectors **v** in W distinct from $\hat{\boldsymbol{y}}$.