Linear Transformation (Sections 1.8, 1.9)

General view: Given an input, the transformation produces an output.
In this sense, a function is also a transformation.

Example. Let A = $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$ and **x** = $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Describe matrix-vector multiplication A**x** in the language of transformation.

$$\boldsymbol{Ax} = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \equiv \boldsymbol{b}$$

Vector **x** is transformed into vector **b** by left matrix multiplication

Definition and terminologies.

Transformation (or function or mapping) T from R^n to R^m is a rule that assigns to each vector $\mathbf{x}$ in R^n a vector $T(\mathbf{x})$ in R^m.

- Notation: $T: R^n \to R^m$
- R^n is the *domain* of T
- R^m is the *codomain* of T
- $T(\mathbf{x})$ is the *image* of vector $\mathbf{x}$
- The set of all images $T(\mathbf{x})$ is the *range* of T
- When $T(\mathbf{x}) = A\mathbf{x}$, A is a $m \times n$ size matrix. Range of T = Span{ column vectors of A} (HW1.8.7)

See class notes for other examples.

Linear Transformation --- Existence and Uniqueness Questions (Section 1.9)

Definition 1: T: $R^n \to R^m$ is **onto** if each **b** in R^m is *the image of* ***at least one*** **x** in R^n.

• i.e. codomain R^m = range of T

• When solve $T(\mathbf{x}) = \mathbf{b}$ for **x** (or A**x**=**b**, A is the standard matrix), there exists at least one solution (Existence question).

Definition 2: T: $R^n \to R^m$ is **one-to-one** if each **b** in R^m is *the image of* ***at most one*** **x** in R^n.

• i.e. When solve $T(\mathbf{x}) = \mathbf{b}$ for **x** (or A**x**=**b**, A is the standard matrix), there exists either a unique solution or none at all (Uniqueness question).

See class notes for Example 4.

Linear Transformation --- Existence and Uniqueness Questions (Section 1.9) cont'd

Facts to determine whether linear transformation $T: R^n \to R^m$ **onto** or **one-to-one** or both.

Theorem 11: $T: R^n \to R^m$ is **one-to-one** if and only if $T(\mathbf{x}) = \mathbf{0}$ has only trivial solution.

- Comment: $A\mathbf{x} = \mathbf{0}$ (A is the standard matrix of T) $\to$ Columns of A are linearly independent.

Theorem 12: Let $T: R^n \to R^m$ be a linear transformation and let A be the standard matrix. Then:

a. T maps R^n **onto** R^m if and only if columns of A *spans* R^m.

b. T is **one-to-one** if and only if the columns of A are *linearly independent.*

- Comment: (a) is true by Theorem 4. (b) is true by Theorem 11.

See class notes for other Example 5.

Matrix Operations (Section 2.1)

Matrix notation. Let A be a m × n matrix. Let $\mathbf{a_1}$, … , $\mathbf{a_n}$ be columns (or column vectors) of A.
A=[$\mathbf{a_1}$, $\mathbf{a_2}$ … , $\mathbf{a_n}$]

Denote a_{ij} the entry at the *i*th row and *j*th column of A.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a_1} & & \mathbf{a_j} & & \mathbf{a_n} \end{array}$$

Matrix-matrix addition. Let A and B be m × n matrices. (A + B by adding corresponding entries.)

Example 1. $A=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ $C=\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$

$$A+B=\begin{bmatrix} 4+1 & 0+1 & 5+1 \\ -1+3 & 3+5 & 2+7 \end{bmatrix}=\begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

A and C (or B and C) can not be added together because of different sizes.

Scalar multiple of matrix. Let A be a m × n matrix, *r* be a number. *r*A is the scalar multiple by *r* times each entry of A

Define: -A = (-1)A.

Example 2. $$2B=\begin{bmatrix} 2(1) & 2(1) & 2(1) \\ 2(3) & 2(5) & 2(7) \end{bmatrix}=\begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$A-2B=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}-\begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}=\begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

Theorem 2.1. Let A, B and C be matrices of same size. Let *r* and *s* be scalars (numbers).

a. A+B = B+A.
b. (A+B) + C = A + (B + C)
c. A + 0 = A (0 represents matrix whose entries are all zeros)
d. $r(A + B) = rA + rB$
e. $(r+s)A = rA + sA$
f. $r(sA) = (rs)A$

Matrix-matrix multiplication. Let A be $m\times n$ matrix. Let B be $n\times p$ matrix with columns $\mathbf{b}_1 , \mathbf{b}_2 , \ldots , \mathbf{b}_p$. Let $\mathbf{x}$ be a vector in R^p (we will see why we choose these sizes for these matrices and the vector soon).

AB is based on $A(B\mathbf{x})$.

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_p\mathbf{b}_p.$$

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_p\mathbf{b}_p) \\ &= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \ldots + x_pA\mathbf{b}_p \\ &= [A\mathbf{b}_1 , A\mathbf{b}_2 , \ldots , A\mathbf{b}_p]\, \mathbf{x} \end{aligned}$$

Definition. Let A be $m\times n$ matrix. Let B be $n\times p$ matrix with columns $\mathbf{b}_1 , \mathbf{b}_2 , \ldots , \mathbf{b}_p$. AB is a $m\times p$ matrix whose columns are $A\mathbf{b}_1 , A\mathbf{b}_2 , \ldots , A\mathbf{b}_p$

$$AB = A[\mathbf{b}_1 , \mathbf{b}_2 , \ldots , \mathbf{b}_p] = [A\mathbf{b}_1 , A\mathbf{b}_2 , \ldots , A\mathbf{b}_p]$$

See class notes for examples