Matrix operations (Section 2.1) cont'd

Theorem 2.2 Let A be a m × n matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. A(BC) = (AB)C.
b. A(B+C) = AB + AC
c. (B+C)A = BA + CA
d. r(AB) = (rA)B = A(rB) for any scalar r
e. $I_m A = A = A I_n$ I_m and I_n are identity matrices

Important: AB and BA are usually not the same.

Example 7. $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$ Compute AB and BA

Solution:

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

Matrix operations (Section 2.1) cont'd

Definition. Let A be a m × n matrix. The **transpose** of A is a n × m matrix, denoted by A^T, whose columns are from the corresponding rows of A.

Example 8.

$$A = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$$

Compute A^T

Solution:

$$A^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$$

Theorem 2.3. Let A and B be matrices with appropriate sizes for which the indicated sums and products are defined.

a. $(A^T)^T = A$
b. $(A+B)^T = A^T + B^T$
c. $(rA)^T = rA^T$ for any scalar r
d. $(AB)^T = B^T A^T$

Powers of a matrix. Let A be a n × n matrix. A^k denotes the product of "k" copies of A. For example: $A^2 = AA$; $A^3 = AAA$.

Inverse of a matrix (Section 2.2)

Definition Let A be a n × n matrix. A is **invertible** if there is a n × n matrix, denoted by A^{-1} such that $A^{-1}A = I$; $AA^{-1} = I$. Here I is the n × n identity matrix.

Example 1.

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix},$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 2.4 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is **invertible** and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, the A is **not invertible**.

Definition. $ad - bc$ is the determinant of A, denoted by $\det A = ad - bc.$

Theorem 2.5. If A is an **invertible** n × n matrix , $A\mathbf{x} = \mathbf{b}$ has the **unique solution** $\mathbf{x} = A^{-1}\mathbf{b}$ for any **b** in R^n.

Inverse of a matrix (Section 2.2) cont'd

Example 4. solve the system $\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases}$

Solution: Let's define $\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. Then det A = 3(6) – 5(4) = -2

$$\boldsymbol{A}^{-1} = \frac{1}{(-2)} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Theorem 2.6.

a. Let A be an invertible matrix, then $(A^{-1})^{-1} = A$.

b. Let A and B be both invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

c. Let A be an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$.

Inverse of a matrix (Section 2.2) cont'd

Definition. An **elementary matrix** is the matrix obtained by performing a single elementary row operations on an identity matrix.

Example: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. E_1 by row3+ (-2)row1.
2. E_2 by exchange row2, row3.
3. E_3 by (5)row3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Next: **Two important facts** associated with elementary matrix.

Inverse of a matrix (Section 2.2) cont'd

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

1. Do elementary row operation: row3+ (-2)row1.
2. Write down the corresponding elementary matrix E
3. Compute EA

Solution: 1. The result is $\begin{bmatrix} a & b & c \\ d & e & f \\ g-2a & h-2b & i-2c \end{bmatrix}$

2. The corresponding elementary matrix is $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

3. $EA = \begin{bmatrix} a & b & c \\ d & e & f \\ g-2a & h-2b & i-2c \end{bmatrix}$

Fact 1. If an elementary row operation is performed on a $m \times n$ matrix A, the result is EA, where E is the $m \times m$ elementary matrix obtained by the same row operation on identity matrix I_m.

Inverse of a matrix (Section 2.2) cont'd

Example: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Elementary matrix E_1 is obtained by row3+ (-2)row1.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

To transform E_1 into I, do row3 + (2)row1. The corresponding elementary matrix is

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. By **Fact 1**, $E_1E = I$ (we can similarly show $EE_1 = I$). Thus $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Fact 2. Each elementary matrix E is invertible. The inverse of E is the elementary matrix that transforms E back to I.