Characterizations of invertible matrix (Section 2.3)

Theorem 2.8 (Invertible matrix theorem) Let A be a $n \times n$ matrix. Then the following statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is **one-to-one**.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in R^n (This could be stated as " $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in R^n ").
h. The columns of A span R^n.
i. The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps R^n **onto** R^n
j. There is a $n \times n$ matrix C such that $CA = I$ (I is the $n \times n$ identity matrix).
k. There is a $n \times n$ matrix D such that $AD = I$ (I is the $n \times n$ identity matrix).
i. A^T is an invertible matrix.

Example. $$A = \begin{bmatrix} 2 & 7 \\ 0 & 3 \end{bmatrix}$$

Characterizations of invertible matrix (Section 2.3)cont'd

Example 1. Use **Theorem 2.8** to decide if A is invertible:

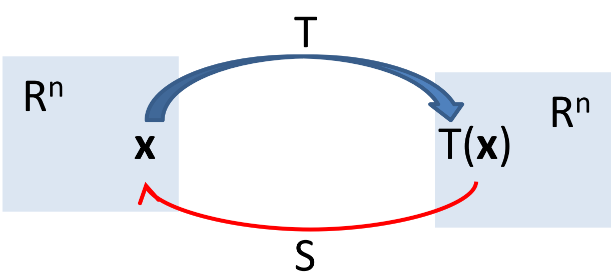
$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 5 & -1 & 9 \end{bmatrix}$$

Solution. Use **Theorem 2.8.c** and row reduction alg.

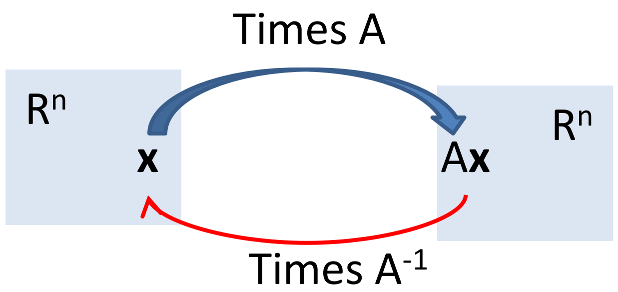
$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Invertible Linear transformation. A linear transformation T : $R^n \rightarrow R^n$ is **invertible** if there exists a mapping (which is also a linear transformation as we shall see later) S : $R^n \rightarrow R^n$ such that:

1. S(T(**x**)) = **x** for all **x** in R^n
2. T(S(**x**)) = **x** for all **x** in R^n



Characterizations of invertible matrix (Section 2.3)cont'd



Theorem 2.9 Let $T : R^n \rightarrow R^n$ be a linear transformation and let A be the standard matrix for T. Then T is **invertible** if and only if A is an invertible matrix. In that case, the linear transformation $S : R^n \rightarrow R^n$ given by $S(A) = A^{-1}\mathbf{x}$ is the unique function satisfying:

1. $S(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in R^n
2. $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in R^n

Example 2. Let $T : R^n \rightarrow R^n$ be an one-to-one linear transformation. (1) Is T invertible? (2) Does T map R^n onto R^n?

Solution: Let A be the standard matrix of T.

(1) Columns of A are linearly independent (by Theorem 1.12) → A is invertible (by Theorem 2.8) → T is invertible.

(2) A is invertible → T is onto (by Theorem 2.8 as well).