Vector space (Section 4.1)

Examples of generalized vectors: $$\begin{bmatrix}1\\2\\4\end{bmatrix};\quad 1+\sin(t/2);\quad 1+3t+5t^3;\quad \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Example1. For n ≥ 0, the set P_n of polynomials of degree at most *n* consists of all polynomials of the form: $\boldsymbol{p}(t) = a_0 + a_1t + a_2 t^2 + \ldots + a_n t^n$. Here a_0, a_1, a_2, … a_n and variable *t* are real numbers.

The **degree** of $\boldsymbol{p}$ is the highest power of *t* in $a_0 + a_1t + a_2 t^2 + \ldots + a_n t^n$ whose coefficient is not zero.

The **degree** of $\boldsymbol{p}(t) = a_0 \neq 0$ is zero.

$\boldsymbol{p}(t) = 0$ is called the zero polynomial.

Example2. The set P_5 of polynomials of degree at most *5* consists of all polynomials of the form: $\boldsymbol{p}(t) = a_0 + a_1t + a_2 t^2 + \ldots + a_5 t^5$. Here a_0, a_1, a_2, … a_5 and variable *t* are real numbers.

$P_5 = \{ 0, 2, 1 + t, 2 + t^2, 3 - 5t + 10t^4, 9 + 10t^3 + 4t^5, \ldots\}$.

Polynomial $40t^6$ *is not in* P_5.

Example 3. All real-valued functions defined on a set D (the set of real numbers or some interval on the real time) form a set V.

Let D = R, V = $\{0, 1 + t, \sin(2t) + 5t, \ldots\}$

Vector space (Section 4.1 cont'd)

Definition. A **vector space** is a nonempty set V of objects, on which two operations, "addition" and "multiplication by scalars (real numbers)" are defined. Moreover, **these two operations have to satisfy** the following 10 axioms.

For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and scalars c and d:

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is **zero** vector $\mathbf{0}$ in V, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. $-\mathbf{u}$ is called the negative of $\mathbf{u}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u}$
8. $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Note: **zero** vector is unique. $-\mathbf{u}$ is unique for each $\mathbf{u}$ in V.

$0\mathbf{u} = \mathbf{0}$

$c\mathbf{0} = \mathbf{0}$

$-\mathbf{u} = (-1)\mathbf{u}$

Example. The set R^n is a vector space.