1. Augmented matrix

$$\begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 4 \\ 1 & 1 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & -9 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 4 \\ x_3 = 0 \end{cases} \quad \text{(b)}$$

2. Key: is there free variable(s)?

$$\begin{bmatrix} -2 & 8 & 1 & 0 \\ 3 & 5 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & 1 & 0 \\ 0 & 17 & -9/2 & 0 \end{bmatrix}$$

one free variable → what's the parametric vector form of soln? (A).

3. Rewrite $\begin{bmatrix} x_1+2x_2-3x_3 \\ -x_1+3x_2-7x_3 \end{bmatrix}$ into $A\vec{x}$

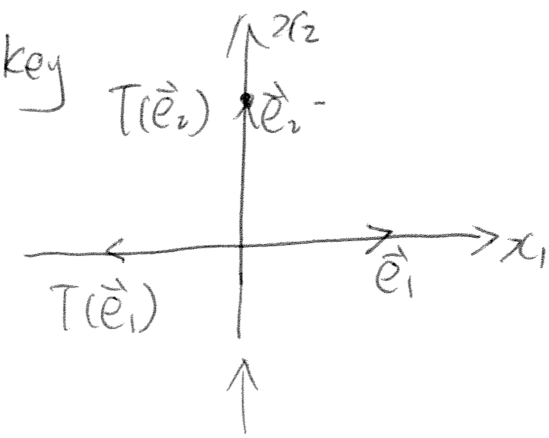
Thus $= \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ -7x_3 \end{bmatrix}$

$= x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ -7 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & -3 \\ -1 & 3 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (a).

4. key: solve $x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by definition. However, we have a zero vector. By Thm 9, $\Rightarrow$ (a).

5. key



$T(\vec{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\boxed{A = [\,T(\vec{e}_1)\ \ T(\vec{e}_2)\,]}$

reflect through x_2-axis (Thm 10).

6. What's standard matrix A?

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$ (use approach for solving Prob. 3)

Consider Thm 12 (also study Thm 11).
Columns of A are linearly independent (why?)
(They are not scalar multiple of each other).

Ans: (a).

7. $A^3 = AAA$ (matrix multiplication).

8. Key: $\boxed{\vec{u}+\vec{v} \text{ has to stay in } H.}$

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

& $u_1^2 - u_2^2 = 0$, $v_1^2 - v_2^2 = 0$

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$$

Does $(u_1+v_1)^2 - (u_2+v_2)^2 \overset{?}{=} 0$?
In general, NO.

(9) key: Find parametric vector form of soln to $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 1 & 9 & 0 \\ 2 & 3 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 9 & 0 \\ 0 & 1 & -11 & 0 \end{bmatrix}$$

free variable = x_3

# of free variable = dim(Nul A). (Why?)

(10) A has to be $n \times n$. (square matrix)

(11) Solvable ⇒ we $\boxed{\text{can not}}$ have "0 = const." eqn.

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & 2 & 6 & -5 \\ 0 & 0 & 0 & h+4 \end{bmatrix}$$

⇒ $h = -4$.

(12) key: row-reduction to reduce A to echelon form, count pivot columns.

(13) Use $[A|I] \sim [I|A^{-1}]$

(14) Solve $c_1\begin{bmatrix}1\\4\end{bmatrix} + c_2\begin{bmatrix}2\\9\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$ for c_1 & c_2

$$\begin{bmatrix}1 & 2 & 3\\4 & 9 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3\\0 & 1 & -10\end{bmatrix}$$

$$c_2 = -10$$

$$c_1 = 3 - 2c_2 = 23$$

$$[\vec{x}]_B = \begin{bmatrix}23\\-10\end{bmatrix}$$

(15) key : reduce to echelon form,
Columns of A crsp. to pivot columns
in echelon form make the basis
for Col A.