Solution key to Set 1.

(1.) Notice A is matrix with orthonormal columns.
Then $A^TA = I$. Therefore $A^{-1} = A^T$

Remark: Study properties of matrix with orthonormal columns

(2) $\text{Dim}(W) = 1$. Therefore Dim (orth. complement of W) $= 2$.
So possible solns are (a), (c), (e).
since $\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\0\end{pmatrix} = 0$, $\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-3\\0\\1\end{pmatrix} = 0$. Answer is (a).

(3) Eqn is 1st order linear eqn. Use integrating factor
$y' + \frac{y}{5} = 9.8$, Therefore $P(t) = \frac{1}{5}$
Integrating factor $\mu(t) = e^{\int P(t)dt} = e^{t/5}$.
Multiply $e^{t/5}$ to every term of the DE, we obtain

$$\frac{d}{dt}(e^{t/5}y) = 9.8e^{t/5}$$

$$e^{t/5}y = 49e^{t/5} + C$$

$$y = 49 + Ce^{-t/5}$$

Use I.C. $y(0) = 50 \Rightarrow C = 1$

Therefore, soln is $y = 49 + e^{-t/5}$

4 Notice $v_1 \cdot v_2 = 0$, which implies v_1 and v_2 are orthogonal.

Then $\text{Proj}_W y = \frac{v_1 \cdot y}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot y}{v_2 \cdot v_2} v_2$

5 This is solved by Gram-Schmidt process.

Also, notice $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$ are orthogonal.

Let $v_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$. $v_3 = \begin{pmatrix}1\\3\\1\\7\end{pmatrix}$

Then $u_3 = v_3 - \frac{v_1 \cdot v_3}{v_1 \cdot v_1} v_1 - \frac{v_2 \cdot v_3}{v_2 \cdot v_2} v_2$

6 First use Gram-Schmidt process to contruct a orthogonal basis. Then normalize vectors in the basis to form an orthonormal basis.

7 Compute $\text{Proj}_u y = \frac{u \cdot y}{u \cdot u} u = \frac{40}{20}\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}$

8 Solve $A^T A \vec{\hat{x}} = A^T \vec{b}$ for $\vec{\hat{x}}$.

9 Change the DE to $y' + \frac{2}{t} y = 4t$.

This is a 1^{st} order linear ODE. Use integrating factor method to solve.

The integrating factor is $\mu(t) = e^{\int \frac{2}{t} dt} = t^2$

Then $\frac{d(t^2 y)}{dt} = 4t^3$

$t^2 y = t^4 + c$

$y = t^2 + \frac{c}{t^2}$, where c is const. of Integration

use $y(1) = 3$. $\Rightarrow c = 2$

Thus the soln is $y(t) = t^2 + \frac{2}{t^2}$

10 $y\,dy = \frac{x^2}{1+x^3} dx$

$\frac{y^2}{2} = \frac{1}{3} \ln|1+x^3| + c.$

11 Let $\vec{z} = \vec{y} - \text{Proj}_W \vec{y}$.

Then the distance is $\|\vec{z}\|$.

This is a problem related to orthogonal decomposition Thm.

12 This is a bank problem.

Let the value of investment at time "t" be $S(t)$.

Then $\begin{cases} \frac{ds}{dt} = rs \\ S(0) = S_0 \end{cases}$

The soln to this IVP is $S(t) = S_0 e^{rt}$.

Now $2S_0 = S_0 e^{r10}$, solve for r.

$r = \frac{\ln 2}{10}$.

13 Char. eqn $r^2 - 3r + 2 = 0$

$r_1 = 2, \ r_2 = 1$

General soln is $y(t) = C_1 e^{2t} + C_2 e^t$

$y(0) = 0 \Rightarrow 0 = C_1 + C_2$

$\left.\begin{array}{l} y'(0) = 1 \\ y'(t) = 2C_1 e^{2t} + C_2 e^t \end{array}\right\} \Rightarrow 1 = 2C_1 + C_2$

$C_1 = 1, \ C_2 = -1.$

Soln to IVP is: $y(t) = e^{2t} - e^t$.

$\phi(\ln 2) = e^{2\ln 2} - e^{\ln 2} = 4 - 2 = 2.$

(14) Use separable eqn method.
Also notice this is a logistic model.

$$\frac{dy}{(1-y)y} = \frac{dt}{2}$$

$$\left[\frac{1}{1-y} + \frac{1}{y}\right] dy = \frac{dt}{2}$$

$$\ln|y| - \ln|1-y| = \frac{t}{2} + k,$$ where k is const. of integration.

$$\frac{y}{y-1} = ce^{t/2}$$

(With this init. condition $y(0)=4$, $y(t) > 1$. Here 1 is the saturation level.)

Use I.C. $y(0)=4$

$$c = 4/3$$

$$y = \frac{4e^{t/2}}{4e^{t/2}-3}$$

(15) Try integrating factor being function of x, call it $\mu(x)$.

$$\mu(x)(xy+x+1) + \mu(x)(x^2+xy)y' = 0$$

$$M = \mu(x)(xy+x+1)$$

$$N = \mu(x)(x^2+xy)$$

$$M_y = \mu(x)x, \quad N_x = \mu'(x^2+xy) + \mu(2x+y)$$

$M_y = N_x$ implies

$$0 = \mu'(x)x(x+y) + \mu(x)(x+y)$$

$$x\mu' = -\mu$$

$$\Rightarrow \quad \mu(x) = \frac{1}{x}.$$

Then solve $(y+1+\frac{1}{x}) + (x+y)y' = 0$. This eqn is exact.

(16) This is also a least-square problem

First solve $A^T A \vec{\hat{x}} = A^T \vec{b}$ for $\vec{\hat{x}}$.

The smallest value is $\|A\vec{\hat{x}} - \vec{b}\|$ (why?).

($A\vec{\hat{x}}$ is the best approximation to $\vec{b}$ in Col(A)).