Numerical Analysis
First midterm
Fall 2006

Name:____

To receive credit you must show your work.

Problems

Problem 1 (20 points)
Let the base be B = 10, and the mantissa length be t = 3. Perform the following computations and compute the relative error in each case:
(i) (100 + 0.4) + 0.4;
(ii) (0.4 + 0.4) + 100.

Soln:
(i) (100 + 0.4) + 0.4 = 100; relative error = |100-100.8|/100.8 $\approx$ 0.00794
(iii) (0.4 + 0.4) + 100 = 101; relative error = |101-100.8|/100.8 $\approx$ 0.00198

Problem 2 (20 points)
Let a sequence x_n be defined inductively by $x_{n+1} = F(x_n)$. Suppose that $x_n \to x$ as $n \to \infty$, $F^{(1)}(x) = 0$, and F is continuously differentiable. Prove that $x_{n+2} - x_{n+1} = o(x_{n+1} - x_n)$.
Soln:
$\lim\limits_{n\to\infty} \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} = \lim\limits_{n\to\infty} \frac{F(x_{n+1}) - F(x_n)}{x_{n+1} - x_n} = \lim\limits_{n\to\infty} F^{(1)}(\xi)$ by the Mean Value Theorem for some ξ between x_n and x_{n+1}.
Since $x_n \to x$ and $x_{n+1} \to x$, therefore $\xi \to x$ and $\lim\limits_{n\to\infty} F^{(1)}(\xi) = F^{(1)}(x) = 0$.

Problem 3 (20 points)
If we compute the root of function $x - \ln x = 2$ within interval $[2, \infty)$ by the bisection method, how many steps are required if the absolute error is less than 0.5×10^{-6}.
Soln:
Let the right end point be b, b is big enough so that the root is within $[2, b]$. (You can plot y = x-2 and y = ln(x). Then you will find that there is only one root within $[2, \infty)$.
Since $| r - c_n | \le 2^{-(n+1)}(b-2) \le 0.5 \times 10^{-6}$.
$\Rightarrow n \ge [\log(b-2) - \log(0.5 \times 10^{-6})]/\log 2 - 1$. (Recall Problem 3.1.4).

Problem 4 (20 points)

Suppose we solve for the positive root of $x^3 - x - 1 = 0$ using the fixed point iteration. There are two iteration functions available: (1). $x = \sqrt[3]{x+1}$, (2). $x = \dfrac{2x^3+1}{3x^2-1}$. Why (1) is inferior to (2) (in terms of the rate of convergence)?

Soln:

These are fixed point iterations. Assume the root is s, s > 0.

For (1), $F(x) = \sqrt[3]{x+1} \Rightarrow F^{(1)}(x) = \dfrac{1}{3(x+1)^{2/3}}$. Since $|F^{(1)}(s)| = \dfrac{1}{3(s+1)^{2/3}} < 1$, $F^{(1)}(x)$ is continuous in the nbhd of s, and $|F^{(1)}(s)| \neq 0$, (1) converges linearly.

For (2), $F(x) = (2x^3+1)/(3x^2-1) \Rightarrow F^{(1)}(x) = \dfrac{6x(x^3-x-1)}{(3x^2-1)^2}$. Since $|F^{(1)}(s)| = 0$, as long as x_0 is close to s, (2) converges quadratically.

(See remark of Theorem 3.4.2).

Problem 5 (20 points)

(a) Find the Doolittle's and Crout's factorization of matrix A = $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 0 & 3 \end{pmatrix}$.

(b) Let $b = (-1,2,3)^T$. Find a solution for Ax=b using Gaussian elimination.

(Hint, you can do (b) first.)

Soln:

(b) If we do (b) first, we will have

$$[A^{(3)} \mid b^{(3)}] = \begin{bmatrix} 1 & 1 & 0 & -1 \\ (1) & 1 & 2 \mid & 3 \\ (2) & (-2) & 7 & 11 \end{bmatrix},$$

$$x = \begin{bmatrix} -6/7 \\ -1/7 \\ 11/7 \end{bmatrix}.$$

(a) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ from Gaussian elimination. This is Doolittle's factorization.

Since $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \Rightarrow \tilde{L} = L\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 7 \end{bmatrix}$.

We have Crout's factorization

$$\widetilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 7 \end{bmatrix}, \widetilde{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 6 (Bonus, 10 points)
Assume $f(r) = 0$, in the nbhd. of r $[r-\delta, r+\delta]$, $f^{(2)}(x)$ is continuous and $f^{(1)}(x) \neq 0$. Let a sequence x_n be generated by the secant method. Suppose $x_{n-1}, x_n \in [r-\delta, r+\delta]$, and x_{n-1}, x_n, r are not equal to each other, prove that $e_{n+1} = r - x_{n+1} = e_n e_{n-1}(-\frac{f^{(2)}(\eta_n)}{2f^{(1)}(\zeta_n)})$, where ξ_n, η_n are between min(x_{n-1}, x_n, r) and max(x_{n-1}, x_n, r).

Soln:

From $x_{n+1} = x_n - \dfrac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$,

$$e_{n+1} = r - x_{n+1} = \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})}$$

$$= e_n e_{n-1} \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{f(x_n) - f(x_{n-1})} \quad \text{, where } G(x) = \begin{cases} \dfrac{f(x) - f(r)}{x - r}, x \neq r \\ f'(r), x = r \end{cases}$$

$$= -e_n e_{n-1} \frac{G(x_n) - G(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Use Mean Value Theorem,

$e_{n+1} = -e_n e_{n-1} \dfrac{G'(\psi_n)}{f'(\varsigma_n)}$, ψ_n, ς_n are between x_{n-1} and x_n. If $\psi_n \neq r$,

$G^{(1)}(\psi_n) = \dfrac{f^{(1)}(\psi_n)(\psi_n - r) - f(\psi_n) + f(r)}{(\psi_n - r)^2} = f^{(2)}(\eta_n)/2$ (Taylor expansion), η_n is between ψ_n and r. Otherwise, if $\psi_n = r$,

$G^{(1)}(\psi_n) = f^{(2)}(r)/2$.

Therefore $e_{n+1} = e_n e_{n-1}(-\dfrac{f^{(2)}(\eta_n)}{2f^{(1)}(\zeta_n)})$.

Problem 7 (Bonus, 10 points)
Prove that for a n by n matrix A, A is nonsingular, if all leading principal minors of A are nonsingular, A has a unique Doolittle's factorization.
Soln:
Assume there are two Doolittle's factorizations A = LU = L*U*.
Since A is nonsingular, $L^{*-1} L = U^* U^{-1} = I$, where I is identity matrix.
Therefore L = L*, U = U*.