Binary numbers are expressed as

$$\ldots b_2 b_1 b_0 . b_{-1} b_{-2} \ldots,$$

where each binary digit, or **bit**, is 0 or 1. The base 10 equivalent to the number is

$$\ldots b_2 2^2 + b_1 2^1 + b_0 2^0 + b_{-1} 2^{-1} + b_{-2} 2^{-2} \ldots.$$

For example, the decimal number 4 is expressed as $(100.)_2$ in base two, and $3/4$ is represented as $(0.11)_2$.

0.2.1 Decimal to binary

The decimal number 53 will be represented as $(53)_{10}$ to emphasize that it is to be interpreted as base 10. To convert to binary, it is simplest to break the number into integer and fractional parts and convert each part separately. For the number $(53.7)_{10} = (53)_{10} + (0.7)_{10}$, we will convert each part to binary and combine the results.

Integer part. Convert decimal integers to binary by dividing by 2 successively and recording the remainders. The remainders, 0 or 1, are recorded by starting at the decimal point (or more accurately, **radix**) and moving away (to the left). For $(53)_{10}$, we would have

$$\begin{aligned} 53 \div 2 &= 26 \text{ R } 1 \\ 26 \div 2 &= 13 \text{ R } 0 \\ 13 \div 2 &= 6 \text{ R } 1 \\ 6 \div 2 &= 3 \text{ R } 0 \\ 3 \div 2 &= 1 \text{ R } 1 \\ 1 \div 2 &= 0 \text{ R } 1. \end{aligned}$$

Therefore, the base 10 number 53 can be written in bits as 110101, denoted as $(53)_{10} = (110101.)_2$. Checking the result, we have $110101 = 2^5 + 2^4 + 2^2 + 2^0 = 32 + 16 + 4 + 1 = 53$.

Fractional part. Convert $(0.7)_{10}$ to binary by reversing the preceding steps. Multiply by 2 successively and record the integer parts, moving away from the decimal point to the right.

$$\begin{aligned} .7 \times 2 &= .4 + 1 \\ .4 \times 2 &= .8 + 0 \\ .8 \times 2 &= .6 + 1 \\ .6 \times 2 &= .2 + 1 \\ .2 \times 2 &= .4 + 0 \\ .4 \times 2 &= .8 + 0 \\ &\vdots \end{aligned}$$

Notice that the process repeats after four steps and will repeat indefinitely exactly the same way. Therefore,

$$(0.7)_{10} = (.1011001100110\ldots)_2 = (.1\overline{0110})_2,$$

where overbar notation is used to denote infinitely repeated bits. Putting the two parts together, we conclude that

$$(53.7)_{10} = (110101.1\overline{0110})_2.$$

0.2.2 Binary to decimal

To convert a binary number to decimal, it is again best to separate into integer and fractional parts.

Integer part. Simply add up powers of 2 as we did before. The binary number $(10101)_2$ is simply $1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = (21)_{10}$.

Fractional part. If the fractional part is finite (a terminating base 2 expansion), proceed the same way. For example,

$$(.1011)_2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \left(\frac{11}{16}\right)_{10}.$$

The only complication arises when the fractional part is not a finite base 2 expansion. Converting an infinitely repeating binary expansion to a decimal fraction can be done in several ways. Perhaps the simplest way is to use the shift property of multiplication by 2.

For example, suppose $x = (0.\overline{1011})_2$ is to be converted to decimal. Multiply x by 2^4, which shifts 4 places to the left in binary. Then subtract the original x:

$$\begin{aligned} 2^4 x &= 1011.\overline{1011} \\ x &= 0000.\overline{1011}. \end{aligned}$$

Subtracting yields

$$(2^4 - 1)x = (1011)_2 = (11)_{10}.$$

Then solve for x to find $x = (.\overline{1011})_2 = 11/15$ in base 10.

As another example, assume that the fractional part does not immediately repeat, as in $x = .10\overline{101}$. Multiplying by 2^2 shifts to $y = 2^2 x = 10.\overline{101}$. The fractional part of y, call it $z = .\overline{101}$, is calculated as before:

$$\begin{aligned} 2^3 z &= 101.\overline{101} \\ z &= 000.\overline{101}. \end{aligned}$$

Therefore, $7z = 5$, and $y = 2 + 5/7$, $x = 2^{-2} y = 19/28$ in base 10. It is a good exercise to check this result by converting $19/28$ to binary and comparing to the original x.

Binary numbers are the building blocks of machine computations, but they turn out to be long and unwieldy for humans to interpret. It is useful to use base 16 at times just to present numbers more easily. **Hexadecimal numbers** are represented by the 16 numerals $0, 1, 2, \ldots, 9, a, b, c, d, e, f$. Each hex number can be represented by 4 bits. Thus $(1)_{16} = (0001)_2$, $(8)_{16} = (1000)_2$, and $(f)_{16} = (1111)_2 = (15)_{10}$. In the