4.4 Composite Numerical Integration

**Motivation:** 1) on large interval, use Newton-Cotes formulas are not accurate.

2) on large interval, interpolation using high degree polynomial is unsuitable because of oscillatory nature of high degree polynomials.

**Main idea:** divide integration interval $[a, b]$ into subintervals and use simple integration rule for each subinterval.

**Example**  

a) Use Simpson’s rule to approximate $\int_0^4 e^x \, dx = 53.59819$.  
b) Divide $[0,4]$ into $[0,1] + [1,2] + [2,3] + [3,4]$. Use Simpson’s rule to approximate $\int_0^1 e^x \, dx$, $\int_1^2 e^x \, dx$, $\int_2^3 e^x \, dx$ and $\int_3^4 e^x \, dx$. Then approximate $\int_0^4 e^x \, dx$ by adding approximations for $\int_0^1 e^x \, dx$, $\int_1^2 e^x \, dx$, $\int_2^3 e^x \, dx$ and $\int_3^4 e^x \, dx$. Compare with accurate value.

**Solution:**

a) $\int_0^4 e^x \, dx \approx \frac{2}{3} (e^0 + 4e^2 + e^4) = 56.76958$.  
Error=$|53.59819 - 56.76958| = 3.17143$

b) $\int_0^4 e^x \, dx = \int_0^1 e^x \, dx + \int_1^2 e^x \, dx + \int_2^3 e^x \, dx + \int_3^4 e^x \, dx \approx \frac{0.5}{3} (e^0 + 4e^{0.5} + e^1) + \frac{0.5}{3} (e^1 + 4e^{1.5} + e^2) + \frac{0.5}{3} (e^2 + 4e^{2.5} + e^3) + \frac{0.5}{3} (e^3 + 4e^{3.5} + e^4) = 53.61622$  
Error=$|53.59819 - 53.61622| = 0.01807$

b) is much more accurate than a).
Let $f \in C^2[a,b], h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \cdots, n$.

On each subinterval $[x_{j-1}, x_j]$, for $j = 1, \cdots, n$, apply Trapezoidal rule:

\[
\int_a^b f(x)\,dx = \left[ \frac{h}{2} (f(x_0) + f(x_1)) - \frac{h^3}{12} f''(\xi_1) \right] + \left[ \frac{h}{2} (f(x_1) + f(x_2)) - \frac{h^3}{12} f''(\xi_2) \right] + \cdots \\
+ \left[ \frac{h}{2} (f(x_{n-1}) + f(x_n)) - \frac{h^3}{12} f''(\xi_n) \right] = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{h^3}{12} \sum_{j=1}^{n} f''(\xi_j)
\]
**Theorem 4.5** Let $f \in C^2[a,b], h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \cdots, n$. There exists a $\mu \in (a,b)$ for which **Composite Trapezoidal rule** with its error term is

$$
\int_a^b f(x)dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} h^2 f''(\mu)
$$

Error term

**Composite Simpson’s rule**

Let $f \in C^2[a,b], \ n \textbf{ be an even integer}, \ h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \cdots, n$.

On each consecutive pair of subintervals, for instance $[x_0, x_2], [x_2, x_4]$, and $[x_{2j-2}, x_{2j}]$ for each $j = 1, \cdots, n/2$, apply Simpson’s rule:
Theorem 4.4 Let $f \in C^4[a, b]$, $n$ be even integer, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \cdots, n$. There exists a $\mu \in (a, b)$ for which Composite Simpson’s rule with its error term is

\[
\int_a^b f(x) \, dx = \frac{h}{3} \left[ f(a) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4 \sum_{j=1}^{(n/2)} f(x_{2j-1}) + f(b) \right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)
\]

Error Term
Theorem 4.6 Let \( f \in C^2[a, b] \), \( n \) be even, \( h = \frac{b-a}{n+2} \), and \( x_j = a + (j + 1)h \) for each \( j = -1, 0, \ldots, n, n + 1 \). There exists a \( \mu \in (a, b) \) for which Composite Midpoint rule with its error term is

\[
\int_a^b f(x)dx = 2h \sum_{j=0}^{\left(\frac{n}{2}\right)} f(x_{2j}) + \frac{b-a}{6}h^2f''(\mu)
\]

Error Term