5.3 High-Order Taylor Methods

Consider the IVP

\[ y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \beta. \]

**Definition:** The difference method

\[ w_0 = \beta \]

\[ w_{i+1} = w_i + h \phi(t_i, w_i), \quad \text{for each } i = 0, 1, 2, \ldots, N - 1, \quad \text{with step size } h = \frac{b - a}{N} \]

has **Local Truncation Error**

\[ \tau_{i+1}(h) = \frac{y_{i+1} - \left( y_i + h \phi(t_i, y_i) \right)}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i) \quad \text{for each } i = 0, 1, 2, \ldots, N - 1. \]

Note: \( y_i := y(t_i) \) and \( y_{i+1} := y(t_{i+1}). \)

**Geometric view of local truncation error**

[Diagram showing the Euler approximation to \( dy/dt = y(t^{2+1}, t=0.5) \) with approximate and exact lines, illustrating the local truncation error \( \tau_{i+1}(h)h \) and \( y_{i+1} - w_{i+1} \).]
Example. Analyze the local truncation error of Euler’s method for solving
\[ y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \beta. \]
Assume \(|y''(t)| < M\) with \(M > 0\) constant.

Solution:
\[
\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + hf(t_i,y_i))}{h} = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) = \frac{y(t_i + hf(t_i,y_i)) + \frac{h^2}{2} y'''(\xi_i) - y_i}{h} - f(t_i, y_i)
\]
with \(\xi_i \in (t_i, t_{i+1})\).
\[
\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i).
\]
Thus \(|\tau_{i+1}(h)| \leq \frac{h}{2} M\).
So the local truncation error in Euler’s method is \(O(h)\).

Consider the IVP
\[ y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \beta. \]

Compute \(y'', y^{(3)}\) ….
First, by IVP: \(y'' = f'(t, y(t))\)
\[
y^{(3)}(t) = f''(t, y(t))
\]
\[
\vdots
\]
\[
y^{(k)}(t) = f^{(k-1)}(t, y(t))
\]
Second, by chain rule:
\[
y'' = \frac{dy'}{dt} = \frac{df(t,y(t))}{dt} = \frac{\partial f}{\partial t}(t,y(t)) + \frac{\partial f}{\partial y}(t,y(t)) \cdot y'(t) = \frac{\partial f}{\partial t}(t,y(t)) + \frac{\partial f}{\partial y}(t,y(t)) \cdot f(t,y(t))
\]
\[
\vdots
\]

Derivation of higher-order Taylor methods
Consider the IVP
\[ y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \beta, \quad \text{with step size } h = \frac{b-a}{N}, \quad t_{i+1} = a + ih. \]
Expand \( y(t) \) in the \( n \)th Taylor polynomial about \( t_i \), evaluated at \( t_{i+1} \)

\[
y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)
\]

\[
= y(t_i) + hf(t_i,y(t_i)) + \frac{h^2}{2} f'(t_i,y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i,y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i,y(\xi_i))
\]

for some \( \xi_i \in (t_i,t_{i+1}) \). Delete remainder term to obtain the \( n \)th Taylor method of order \( n \).

Denote \( T^{(n)}(t_i,w_i) = f(t_i,w_i) + \frac{h}{2} f'(t_i,w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i,w_i) \)

**Taylor method of order \( n \)**

\[
\begin{align*}
w_0 &= \beta \\
w_{i+1} &= w_i + hT^{(n)}(t_i,w_i) & \text{for each } i = 0, 1, 2, \ldots, N-1.
\end{align*}
\]

Remark: Euler’s method is Taylor method of order one.

**Example 1.** Use Taylor method of orders (a) two and (b) four with \( N = 10 \) to the IVP

\[
y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.
\]

**Solution:**

\[
h = \frac{2-0}{N} = \frac{2-0}{10} = 0.2. \text{ So } t_i = 0 + 0.2i = 0.2i \quad \text{for each } i = 0, 1, 2, \ldots, 10.
\]

(a) \( f'(t,y(t)) = \frac{dy}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t \)

So \( T^{(2)}(t_i,w_i) = f(t_i,w_i) + \frac{h}{2} f'(t_i,w_i) = (w_i - t_i^2 + 1) + 0.1(w_i - t_i^2 + 1 - 2t_i) = 1.1(w_i - t_i^2 + 1) - 0.2t_i \)

The 2\(^{nd}\) order Taylor method is

\[
w_{i+1} = w_i + 0.2\left(1.1(w_i - t_i^2 + 1) - 0.2t_i\right) \quad \text{for each } i = 0, 1, 2, \ldots, 9
\]

Now compute approximations at each time step:
The 4th order Taylor method is

\[ w_0 = 0.5 \]
\[ w_1 = w_0 + 0.2 \left( 1.1 (w_0 - (0)^2 + 1) - 0.2(0) \right) = 0.83; \quad \text{abs. eror: } |w_1 - y_1| = 0.000701 \]
\[ w_2 = w_2 + 0.2 \left( 1.1 (w_2 - (0.2)^2 + 1) - 0.2(0.2) \right) = 1.2158; \quad \text{abs. eror: } |w_2 - y_2| = 0.001712 \]

(b) \[ f'''(t, y(t)) = \frac{d^3}{dt^3} (f) = (y - t^2 + 1 - 2t)' = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \]
\[ f^{(3)}(t, y(t)) = \frac{d^3}{dt^3} (f'''(t, y(t)) = (y - t^2 - 2t - 1)' = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \]

So \[ T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f^{(3)}(t_i, w_i) \]
\[ = (w_i - t_i^2 + 1) + \frac{h}{2} (w_i - t_i^2 + 1 - 2t_i) + \frac{h^2}{6} (w_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24} (w_i - t_i^2 - 2t_i - 1) \]
\[ = \left( 1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24} \right) (w_i - t_i^2) - \left( 1 + \frac{h}{3} + \frac{h^2}{12} \right) (ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24} \]

The 4th order Taylor method is

\[ w_{i+1} = w_i + h \left( \left( 1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24} \right) (w_i - t_i^2) - \left( 1 + \frac{h}{3} + \frac{h^2}{12} \right) (ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24} \right) \]

for each \( i = 0, 1, 2, \ldots, 9. \)

Now compute approximate solutions at each time step:

\[ w_1 = 0.5 + 0.2 \left( \left( 1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24} \right) (0.5 - 0) - \left( 1 + \frac{0.2}{3} + \frac{0.2^2}{12} \right) (0) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24} \right) = 0.8293 \]

abs. eror of 4th order Taylor at \( t_1: \quad \left| w_1 - y_1 \right| = 0.000001 \)

\[ w_2 = 0.8293 + 0.2 \left( \left( 1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24} \right) (0.8293 - 0.2^2) - \left( 1 + \frac{0.2}{3} + \frac{0.2^2}{12} \right) (0.2(0.2)) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24} \right) \]
\[ = 1.214091 \]

abs. eror 4th order Taylor at \( t_2: \quad \left| w_2 - y_2 \right| = 0.000003 \]
Finding approximations at time other than \( t_i \)

**Example.** (Table 5.4 on Page 259). Assume the IVP \( y' = y - t^2 + 1, \ 0 \leq t \leq 2, \ y(0) = 0.5 \) is solved by the 4th order Taylors method with time step size \( h = 0.2. \ w_6 = 3.1799640 \ (t_6 = 1.2), \ w_7 = 3.7324321 \ (t_7 = 1.4). \) Find \( y(1.25). \)

**Solution:**

Method 1: use linear Lagrange interpolation.
\[
y(1.25) \approx \frac{1.25-1.4}{1.2-1.4} w_6 + \frac{1.25-1.2}{1.4-1.2} w_7 = 3.3180810
\]

Method 2: use Hermite polynomial interpolation (more accurate than the result by linear Lagrange interpolation).

First use \( y' = y - t^2 + 1 \) to approximate \( y'(1.2) \) and \( y'(1.4). \)
\[
y'(1.2) = y(1.2) - (1.2)^2 + 1 \approx 3.1799640 - (1.2)^2 + 1 = 2.7399640
\]
\[
y'(1.4) = y(1.4) - (1.4)^2 + 1 \approx 3.7324321 - (1.4)^2 + 1 = 2.7724321
\]

Then use **Theorem 3.9** to construct Hermite polynomial \( H_3(x). \)
\[
y(1.25) \approx H_3(1.25).
\]

Error analysis

**Theorem 5.12**  If Taylor method of order \( n \) is used to approximate the solution to the IVP
\[
y' = f(t, y), \ 0 \leq t \leq b, \ y(a) = \beta
\]
with step size \( h \) and if \( y \in C^{n+1}[a, b], \) then the **local truncation error** is \( O(h^n). \)

**Remark:** \( y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \)
\[
\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)).
\]
\[
y^{(n+1)}(t) = f^{(n)}(t, y(t)) \text{ is bounded by } |y^{(n+1)}(t)| \leq M.
\]
Thus \( |\tau_{i+1}(h)| \leq \frac{h^n}{(n+1)!} M. \)

So the local truncation error in Euler’s method is \( O(h^n). \)
5.4 Runge-Kutta Methods

Motivation: Obtain high-order accuracy of Taylor’s method without knowledge of derivatives of $f(t, y)$.

Theorem 5.13(Taylor’s Theorem in Two Variables) Suppose $f(t, y)$ and partial derivative up to order $n + 1$ continuous on $D = \{(t, y)|a \leq t \leq b, c \leq y \leq d\}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$f(t, y) = P_n(t, y) + R_n(t, y).$$

Here

$$P_n(t, y) = f(t_0, y_0) + \left[ (t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right]$$

$$+ \left[ \frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right]$$

$$+ \left[ \frac{1}{n!} \sum_{j=0}^{n} \left( n - j \right)(t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right]$$

$$R_n(t, y) = \frac{1}{(n + 1)!} \sum_{j=0}^{n+1} \left( n + 1 - j \right)(t - t_0)^{n+1-j}(y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(t_0, y_0)$$

$P_n(t, y)$ is the $n$th Taylor polynomial in two variables.

Derivation of Runge-Kutta method of order two

1. Determine $\alpha_1, \beta_1$ such that

$$a_1 f(t + \alpha_1, y + \beta_1) \approx f(t, y) + \frac{h}{2} f'(t, y) = T^{(2)}(t, y) \text{ with } O(h^2) \text{ error.}$$

Notice $f'(t, y) = \frac{df(t,y(t))}{dt} = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot y'(t) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$

We have $T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y(t)) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$ \hspace{1cm} (1)

2. Expand $a_1 f(t + \alpha_1, y + \beta_1)$ in $1^\text{st}$ degree Taylor polynomial:
3. Match coefficients of equation (1) and (2) gives

\[ a_1 = 1, \quad a_1 \alpha_1 = \frac{h}{2}, \quad a_1 \beta_1 = \frac{h}{2} f(t, y(t)) \]

with unique solution

\[ a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t, y(t)) \]

4. This gives

\[ T^{(2)}(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y(t))\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y(t))\right) \]

with \( R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y(t))\right) = O(h^2) \)

**Midpoint Method (one of Runge-Kutta methods of order two)**

Consider the IVP

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta. \]

with step size \( h = \frac{b-a}{N} \).

\[ w_0 = \beta \]

\[ w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right), \quad \text{for each} \quad i = 0, 1, 2, \ldots, N - 1. \]

Local truncation error is \( O(h^2) \).

**Two stage formula:**

\[ w_0 = \beta \]

\[ k_1 = f(t_i, w_i) \]

\[ k_2 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} k_1\right) \]

\[ w_{i+1} = w_i + hk_2 \]
Example 2. Use the Midpoint method with \( N = 10, h = 0.2, \ t_i = 0.2i \) and \( w_0 = 0.5 \) to solve the IVP
\[
y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.
\]