5.6 Multistep Methods (cont’d)

Example. Derive Adams-Bashforth two-step *explicit* method: Solve the IVP: \( y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \)

Integrate \( y' = f(t, y) \) over \([y_i, y_{i+1}]\)

\[
y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} y'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt
\]

Use \((t_i, y_i)\) and \((t_{i-1}, y_{i-1})\) to form interpolating polynomial \(P_1(t)\) (by Newton backward difference (Page 129)) to approximate \(f(t, y)\).

\[
\int_{t_i}^{t_{i+1}} f(t, y)\,dt = \int_{t_i}^{t_{i+1}} (f(t_i, y_i) + \nabla f(t_i, y_i) \frac{(t - t_i)}{h} + \text{error})\,dt
\]

\[
y_{i+1} - y_i = h \left[ f(t_i, y_i) + \frac{1}{2} (f(t_i, y_i) - f(t_{i-1}, y_{i-1})) \right] + \text{Error}
\]

where \( h = t_{i+1} - t_i \), and the backward difference \( \nabla f(t_i, y_i) = hf[t_i, t_{i-1}] = (f(t_i, y_i) - f(t_{i-1}, y_{i-1})) \).

Consequently, Adams-Bashforth two-step *explicit* method is:

\[
w_0 = \alpha, \quad w_1 = \alpha_1
\]

\[
w_{i+1} = w_i + \frac{h}{2} [3f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \quad \text{where } i = 1, 2, \ldots N - 1.
\]

**Local Truncation Error.** If \( y(t) \) solves the IVP \( y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \) and

\[
w_{i+1} = a_m w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m}
\]

\[
h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})],
\]

the local truncation error is

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_m y(t_i) + a_{m-2} y(t_{i-1}) + \cdots + a_0 y(t_{i+1-m})}{h} - [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_i))]
\]

**NOTE:** the local truncation error of a \( m \)-step *explicit* step is \( O(h^m) \).

the local truncation error of a \( m \)-step *implicit* step is \( O(h^{m+1}) \).
**Predictor-Corrector Method**

*Motivation:* (1) Solve the IVP $y' = e^y, \quad 0 \leq t \leq 0.25, \quad y(0) = 1$ by the three-step Adams-Moulton method. 

**Solution:** The three-step Adams-Moulton method is

$$w_{i+1} = w_i + \frac{h}{24} \left[ 9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}} \right] \quad Eq. (1)$$

Eq. (1) can be solved by Newton’s method. However, this can be quite computationally expensive.

(2) combine explicit and implicit methods.

**4th order Predictor-Corrector Method**

(we will combine 4th order Runge-Kutta method + 4th order 4-step explicit Adams-Bashforth method + 4th order three-step Adams-Moulton implicit method)

**Step 1:** Use 4th order Runge-Kutta method to compute $w_0, w_1, w_2$ and $w_3$. 

**Step 2:** For $i = 3, 5, ..., N$

(a) Predictor sub-step. Use 4th order 4-step explicit Adams-Bashforth method to compute a predicted value

$$w_{i+1,p} = w_i + \frac{h}{24} \left[ 55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3}) \right]$$

(b) Correction sub-step. Use 4th order three-step Adams-Moulton implicit method to compute a correction (the approximation at $i + 1$ time step)

$$w_{i+1} = w_i + \frac{h}{24} \left[ 9f(t_{i+1}, w_{i+1,p}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2}) \right]$$
5.10 Stability

Consistency and Convergence

**Definition.** A one-step difference equation with local truncation error $\tau_i(h)$ is said to be **consistent** if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| = 0$$

**Definition.** A one-step difference equation is said to be **convergent** if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| = 0$$

where $y(t_i)$ is the exact solution and $w_i$ is the approximate solution.

**Example.** To solve $y' = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$. Let $|y''(t)| \leq M$, $an \ f(t, y)$ be continuous and satisfy a Lipschitz condition with Lipschitz constant $L$. Show that Euler’s method is consistent and convergent.

Solution:

$$|\tau_{i+1}(h)| = \frac{h}{2} y''(\xi_i) \leq \frac{h}{2} M$$

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| \leq \lim_{h \to 0} \frac{h}{2} M = 0$$

Thus Euler’s method is consistent.

By Theorem 5.9,

$$\max_{1 \leq i \leq N} |w_i - y(t_i)| \leq \frac{Mh}{2L} [e^{L(b-a)} - 1]$$

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| \leq \lim_{h \to 0} \frac{Mh}{2L} [e^{L(b-a)} - 1] = 0$$

Thus Euler’s method is convergent.

The rate of convergence of Euler’s method is $O(h)$. 
Stability

Motivation: How does round-off error affect approximation? To solve IVP $y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$ by Euler’s method. Suppose $\delta_i$ is the round-off error associated with each step.

$$u_0 = \alpha + \delta_0$$
$$u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1} \quad \text{for each} \quad i = 0, 1, \ldots, N - 1.$$  

Then $|u_i - y(t_i)| \leq \frac{1}{L} \left( \frac{hM}{2} + \frac{\delta}{h} \right) \left[ e^{L(t_i-a)} - 1 \right] + |\delta_0| e^{L(t_i-a)}$. Here $|\delta_i| < \delta$.

$$\lim_{h \to 0} \left( \frac{hM}{2} + \frac{\delta}{h} \right) = \infty.$$  

Stability: small changes in the initial conditions produce correspondingly small changes in the subsequent approximations.

Convergence of One-Step Methods

Theorem. Suppose the IVP $y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$ is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + h\phi(t_i, w_i, h) \quad \text{where} \quad i = 0, 2, \ldots, N.$$  

Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in $w$ with constant $L$ on $D$, then

$$D = \{(t, w, h) | \quad a \leq t \leq b, -\infty < w < \infty, 0 \leq h \leq h_0\}.$$  

1. The method is stable;
2. The method is convergent if and only if it is consistent:

$$\phi(t, w, 0) = f(t, y), \quad \text{for all} \quad a \leq t \leq b$$

3. If $\tau$ exists s.t. $|\tau_i(h)| \leq \tau(h)$ when $0 \leq h \leq h_0$, then

$$|w_i - y(t_i)| \leq \frac{\tau(h)}{L} e^{L(t_i-a)}.$$