7.1 Norms of Vectors and Matrices

Column vector:
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
, or $\mathbf{x} = [x_1, x_2, \dots, x_n]^t$.

Motivation: Consider to solve the linear system

$$3.3330x_1 + 15920x_2 - 10.333x_3 = 15913$$

$$2.2220x_1 + 16.710x_2 + 9.6120x_3 = 28.544$$

$$1.5611x_1 + 5.1791x_2 + 1.6852x_n = 8.4254$$

by Gaussian elimination with 5-digit rounding arithmetic and partial pivoting. The system has exact solution $\mathbf{x} = [x_1, x_2, x_3]^t = [1,1,1]^t$. The approximate solution is $\tilde{\mathbf{x}} = [1.2001, 0.99991, 0.92538]^t$. How to quantify the approximation error?

Definition. A vector norm on \mathbb{R}^n is a function, $|| \cdot ||$, from \mathbb{R}^n to \mathbb{R} with the properties:

- (i) $||\mathbf{x}|| \ge 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- (ii) $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
- (iii) $||\alpha x|| = |\alpha|||x||$ for all $\alpha \in R$ and $x \in R^n$
- (iv) $||x + y|| \le ||x|| + ||y||$ for all $x, y \in \mathbb{R}^n$

Definition. The **Euclidean norm** l_2 and the **infinity norm** l_{∞} for the vector $\mathbf{x} = [x_1, x_2, ..., x_n]^t$ are defined by

$$||\mathbf{x}||_2 = \{\sum_{i=1}^n x_i^2\}^{1/2}$$

and

$$||\mathbf{x}||_{\infty} = \max_{1 \le i \le n} |x_i|$$

Example. Compute l_2 norm and l_{∞} norm of the vector $\mathbf{x} = [-1, 1, -2]^t$.

Theorem 7.3 Cauchy-Schwarz Inequality for Sums. For each $\mathbf{x} = [x_1, x_2, ..., x_n]^t$ and $\mathbf{y} = [y_1, y_2, ..., y_n]^t$ in \mathbb{R}^n ,

$$\boldsymbol{x}^{t}\boldsymbol{y} = \sum_{i=1}^{n} x_{i} y_{i} \leq \left\{\sum_{i=1}^{n} x_{i}^{2}\right\}^{\frac{1}{2}} \left\{\sum_{i=1}^{n} y_{i}^{2}\right\}^{\frac{1}{2}} = \left||\boldsymbol{x}|\right|_{2} \cdot \left||\boldsymbol{y}|\right|_{2}$$

Remark: $||x + y||_2 \le ||x||_2 + ||y||_2$

Definition. The **distance** between two vectors $\mathbf{x} = [x_1, x_2, ..., x_n]^t$ and $\mathbf{y} = [y_1, y_2, ..., y_n]^t$ is the norm of the difference of the vectors. The l_2 and l_{∞} distances are:

$$||\mathbf{x} - \mathbf{y}||_{2} = \{\sum_{i=1}^{n} (x_{i} - y_{i})^{2}\}^{1/2}$$
$$||\mathbf{x} - \mathbf{y}||_{\infty} = \max_{1 \le i \le n} |x_{i} - y_{i}|$$

Example.

$$\begin{array}{l} 3.3330x_1 + 15920x_2 - 10.333x_3 = 15913\\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 = 28.544\\ 1.5611x_1 + 5.1791x_2 + 1.6852x_n = 8.4254 \end{array}$$

has exact solution $\mathbf{x} = [x_1, x_2, x_3]^t = [1,1,1]^t$. The Gaussian elimination with 5-digit rounding arithmetic and partial pivoting produces approximate solution $\tilde{\mathbf{x}} = [1.2001, 0.99991, 0.92538]^t$. Determine l_2 and l_{∞} distances between exact and approximate solutions.

Solution:

$$||\mathbf{x} - \widetilde{\mathbf{x}}||_{\infty} = \max\{|1 - 1.2001|, |1 - 0.99991|, |1 - 0.92538|\} = \max\{0.2001, 0.00009, 0.07462\} = 0.2001$$

$$||\mathbf{x} - \widetilde{\mathbf{x}}||_2 = \sqrt{(1 - 1.2001)^2 + (1 - 0.99991)^2 + (1 - 0.92538)^2} = 0.21356$$

Definition. A sequence $\{x^{(k)}\}_{k=1}^{\infty}$ of vectors in \mathbb{R}^n is said to **converge** to x with respect to the norm $|| \cdot ||$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$||\mathbf{x}^{(k)} - \mathbf{x}|| < \varepsilon$$
, for all $k \ge N(\varepsilon)$.

Theorem 7.6. The sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in \mathbb{R}^n with respect to the norm $|| \cdot ||_{\infty}$ if and only if $\lim_{k \to \infty} x_i^{(k)} = x_i$. **Example**. Show that $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t = (1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin(k))^t$ converges to $x = (1, 2, 0, 0)^t$. **Solution**:

$$\lim_{k \to \infty} x_1^{(k)} = \lim_{k \to \infty} 1 = 1$$
$$\lim_{k \to \infty} x_2^{(k)} = \lim_{k \to \infty} 2 + \frac{1}{k} = 2$$
$$\lim_{k \to \infty} x_3^{(k)} = \lim_{k \to \infty} \frac{3}{k^2} = 0$$
$$\lim_{k \to \infty} x_4^{(k)} = \lim_{k \to \infty} e^{-k} \sin(k) = 0$$

By **Theorem 7.6**, the sequence $\{x^{(k)}\}$ converges to $(1,2,0,0)^t$.

Theorem 7.7. For each $x \in \mathbb{R}^n$, $||x||_{\infty} \leq ||x||_2 \leq \sqrt{n} ||x||_{\infty}$

Remark: All norms in \mathbb{R}^n are equivalent with respect to convergence, that is, if $|| \cdot ||$ and $|| \cdot ||'$ are any two norms on \mathbb{R}^n and $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ converges to \mathbf{x} in $|| \cdot ||$, then $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ converges to \mathbf{x} in $|| \cdot ||'$.

Definition. A matrix norm $|| \cdot ||$ on $n \times n$ matrices is a real-valued function satisfying

- (i) $||A|| \ge 0$
- (ii) ||A|| = 0 if and only if A = 0
- (iii) $||\alpha A|| = |\alpha|||A||$
- (iv) $||A + B|| \le ||A|| + ||B||$
- $(\mathbf{v}) \quad ||AB|| \le ||A||||B||$

The **distance** between $n \times n$ matrices A and B with respect to a matrix norm is ||A - B||.

Theorem 7.9. If $|| \cdot ||$ is a vector norm, the **induced** (or **natural**) **matrix norm** is given by $||A|| = \max_{\substack{||x||=1}} ||Ax||$

Example. $||A||_{\infty} = \max_{||\mathbf{x}||_{\infty}=1} ||A\mathbf{x}||_{\infty}$, the l_{∞} induced norm. $||A||_{2} = \max_{||\mathbf{x}||_{2}=1} ||A\mathbf{x}||_{2}$, the l_{2} induced norm.

Alternative definition: For any vector $\mathbf{z} \neq \mathbf{0}$, the vector $\mathbf{x} = \mathbf{z}/||\mathbf{z}||$ has $||\mathbf{x}|| = 1$. Since $\max_{||\mathbf{x}||=1} ||A\mathbf{x}|| = \max_{\mathbf{z}\neq\mathbf{0}} ||A(\frac{\mathbf{z}}{||\mathbf{z}||})|| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{||A\mathbf{z}||}{||\mathbf{z}||}$, we can alternatively define $||A|| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{||A\mathbf{z}||}{||\mathbf{z}||}$.

Corollary 7.10. For any vector $\mathbf{z} \neq \mathbf{0}$, matrix *A* and induced matrix norm $|| \cdot ||$, $||A\mathbf{z}|| \leq ||A|| \cdot ||\mathbf{z}||$

Theorem 7.11. If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$\left||A|\right|_{\infty} = \max_{1 \le i \le n} \sum_{j=1}^{n} \left|a_{ij}\right|$$

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Example. Determine $||A||_{\infty}$ for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$

7.2 Eigenvalues and Eigenvectors

Definition. If *A* is an $n \times n$ matrix, the **characteristic polynomial** of *A* is $p(\lambda) = \det(A - \lambda I)$.

Definition. If $p(\lambda)$ is the characteristic polynomial of the matrix A, the zeros of $p(\lambda)$ are **eigenvalues** of the matrix A. If λ is an eigenvalue of A and $x \neq 0$ satisfies $(A - \lambda I)x = 0$, then x is an **eigenvector** corresponding to λ .

Geometric interpretation of **eigenvector** x corresponding to λ .

Example. Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$.

Definition. The **spectral radius** $\rho(A)$ of a matrix *A* is defined by $\rho(A) = \max|\lambda|$, where λ is an eigenvalue of *A*.

Remark: For complex $\lambda = a + bj$, we define $|\lambda| = \sqrt{a^2 + b^2}$.

Theorem 7.15. If *A* is an $n \times n$ matrix, then

- (i) $||A||_2 = [\rho(A^t A)]^{1/2}$
- (ii) $\rho(A) \leq ||A||$, for any induced matrix norm $|| \cdot ||$.