Motivation

Suppose $f \in C[a, b]$, find a polynomial $P_n(x)$ of degree at most $n$ to approximate $f$ such that $\int_a^b (f(x) - P_n(x))^2 \, dx$ is a minimum.

Let polynomial $P_n(x)$ be $P_n(x) = \sum_{k=0}^{n} a_k x^k$ which minimizes the error

$$ E \equiv E(a_0, a_1, \ldots, a_n) = \int_a^b \left( f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 \, dx. $$

The problem is to find $a_0, \cdots, a_n$ that will minimize $E$. The necessary condition for $a_0, \cdots, a_n$ to minimize $E$ is $\frac{\partial E}{\partial a_j} = 0$, which gives the normal equations

$$ \sum_{k=0}^{n} a_k \int_a^b x^{j+k} \, dx = \int_a^b x^j f(x) \, dx \quad \text{for } j = 0, 1, \cdots, n $$
Example Find the least squares approximating polynomial of degree 2 for \( f(x) = \sin \pi x \) on \([0,1]\).

Solution Let \( P_2(x) = a_0 + a_1 x + a_2 x^2 \).

\[
\begin{align*}
\begin{cases}
    a_0 \int_0^1 1 \, dx + a_1 \int_0^1 x \, dx + a_2 \int_0^1 x^2 \, dx = \int_0^1 \sin \pi x \, dx \\
    a_0 \int_0^1 x \, dx + a_1 \int_0^1 x^2 \, dx + a_2 \int_0^1 x^3 \, dx = \int_0^1 x \sin \pi x \, dx \\
    a_0 \int_0^1 x^2 \, dx + a_1 \int_0^1 x^3 \, dx + a_2 \int_0^1 x^4 \, dx = \int_0^1 x^2 \sin \pi x \, dx
\end{cases}
\end{align*}
\]

\begin{align*}
\begin{cases}
    a_0 + \frac{1}{2} a_1 + \frac{1}{3} a_2 = \frac{2}{\pi} \\
    \frac{1}{2} a_0 + \frac{1}{3} a_1 + \frac{1}{4} a_2 = \frac{1}{\pi} \\
    \frac{1}{3} a_0 + \frac{1}{4} a_1 + \frac{1}{5} a_2 = \frac{\pi^2-4}{\pi^3}
\end{cases}
\end{align*}

(1)

\[a_0 = -0.050465, \quad a_1 = 4.12251, \quad a_2 = -4.12251.\]
Definition

The set of functions \( \{ \phi_0, \cdots, \phi_n \} \) is said to **linearly independent** on \([a, b]\) if, whenever \( c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0 \), for all \( x \in [a, b] \) we have \( c_0 = c_1 = \cdots = c_n = 0 \). Otherwise the set of functions is said to be **linearly dependent**.

Theorem

Suppose that, for each \( j = 0, 1, \cdots, n \), \( \phi_j(x) \) is a polynomial of degree \( j \). Then \( \{ \phi_0, \cdots, \phi_n \} \) is linearly independent on any interval \([a, b]\).

Theorem

Suppose that \( \{ \phi_0(x), \cdots, \phi_n(x) \} \) is a collection of linearly independent polynomials in \( \prod_n \). Then any polynomial in \( \prod_n \) can be written uniquely as a linear combination of \( \phi_0(x), \cdots, \phi_n(x) \).
Orthogonal Functions

Definition

\( \{ \phi_0, \cdots, \phi_n \} \) is said to be an orthogonal set of functions for the interval \([a, b]\) with respect to the weight function \(w\) if

\[
\int_a^b w(x) \phi_j(x) \phi_k(x) \, dx = \begin{cases} 0, & \text{when } j \neq k, \\ \alpha_k > 0, & \text{when } j = k. \end{cases}
\]

If also \(\alpha_k = 1\) for each \(k = 0, \ldots, n\), the set is orthonormal.

Theorem

If \( \{ \phi_0, \ldots, \phi_n \} \) is orthogonal on \([a, b]\), then the least squares approximation to \(f\) on \([a, b]\) is \(P(x) = \sum_{k=0}^n a_k \phi_k(x)\) where

\[
a_k = \frac{\int_a^b w(x) \phi_k(x) f(x) \, dx}{\int_a^b w(x) \phi_k(x)^2 \, dx} = \frac{1}{\alpha_k} \int_a^b w(x) \phi_k(x) f(x) \, dx.
\]
The set of Legendre polynomials \( \{P_n(x)\} \) is orthogonal on \([-1, 1]\) w.r.t. the weight function \( w(x) = 1 \).

\[
\begin{align*}
P_0(x) &= 1, & \alpha_0 &= 2 \\
P_1(x) &= x, & \alpha_1 &= 2/3 \\
P_2(x) &= x^2 - \frac{1}{3}, & \alpha_2 &= 8/45 \\
P_3(x) &= x^3 - \frac{3}{5}x, & \alpha_3 &= 8/175 \\
P_4(x) &= x^4 - \frac{6}{7}x^2 + \frac{3}{35}, & \alpha_4 &= 128/11025
\end{align*}
\]