1.3 Algorithms and Convergence

Pseudocode

Example. Compute $\sum_{i=1}^{N} x_i$

INPUT $N, x_1, x_2, \ldots, x_N$.

OUTPUT $SUM = \sum_{i=1}^{N} x_i$

Step 1 Set $SUM = 0$. // Initialize accumulator

Step 2 For $i = 1, 2, \ldots N$ *do*

set $SUM = SUM + x_i$. // add next term

Step 3 OUTPUT(SUM);

STOP.

Characterizing Algorithms

Error Growth

Suppose $E_0 > 0$ denotes an initial error, and E_n is the error after n subsequent operations.

1. If $E_n \approx CnE_0$, where C is a const. independent of n: the growth of error is **linear**.
2. If $E_n \approx C^n E_0$, where $C > 1$: the growth of error is **exponential.**

Remark: linear growth is unavoidable; exponential growth must be avoided.

Stability

- Stable algorithm: small changes in the initial data produce small changes in the final result
- Unstable or conditionally stable algorithm: small changes in all or some initial data produce large errors

Rate of convergence for sequences

Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence converging to 0, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with

$$|\alpha_n - \alpha| \le K|\beta_n|, \qquad \text{for large } n,$$

then $\{\alpha_n\}_{n=1}^{\infty}$ is said to converges to α with rate of convergence $O(\beta_n)$, indicated by $\alpha_n = \alpha + O(\beta_n)$.

Typical $\{\beta_n\}_{n=1}^{\infty}$:

$$\beta_n = \frac{1}{n^p} \quad \text{for some } p > 0$$

Rate of convergence for functions

Suppose that $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$.
If a positive constant K exists with

$$|F(h) - L| \le K|G(h)|, \qquad \text{for sufficiently small } h,$$

then $F(h) = L + O\big(G(h)\big)$.

Typical $G(h)$:

$$G(h) = h^P \ \text{ for some } p > 0$$