2.1 The Bisection Method
Basic Idea

• Suppose function $f(x)$ is continuous on $[a, b]$, and $f(a)$, $f(b)$ have opposite signs.
• By the Intermediate Value Theorem (IVT), there must exist an $p$ in $(a, b)$ with $f(p) = 0$.
• Bisect (sub)intervals and apply IVT repeatedly.
The sequence of intervals \( \{(a_i, b_i)\}_{i=1}^{\infty} \) contains the desired root.
• Intervals containing the root: \((a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset (a_4, b_4) \ldots\)

• After \(n\) steps, the interval \((a_n, b_n)\) has the length:

\[
b_n - a_n = \left(\frac{1}{2}\right)^{n-1} (b - a)
\]

• Let \(p_n = \frac{b_n + a_n}{2}\) be the mid-point of \((a_n, b_n)\). The limit of sequence \(\{p_n\}_{n=1}^{\infty}\) is the root.
The Algorithm

INPUT \ a, b; tolerance TOL; maximum number of iterations N0.
OUTPUT solution p or message of failure.
STEP1 Set i = 1;
FA = f(a);
STEP2 While i ≤ N0 do STEPs 3-6.
STEP3 Set p = a + (b-a)/2; // a good way of computing middle point
FP = f(p).
STEP4 IF FP = 0 or (b-a) < TOL then
OUTPUT (p);
STOP.
STEP5 Set i = i +1.
STEP6 IF FP·FA > 0 then
Set a = p;
FA = FP.
else
set b = p;
STEP7 OUTPUT(“Method failed after N0 iterations”);
STOP.
function p=bisection(f,a,b,tol)

while 1
    p=(a+b)/2;
    if p-a<tol, break; end
    if f(a)*f(p)>0
        a=p;
    else
        b=p;
    end
end %while 1
Stopping Criteria

• Method 1: \( |p_n - p_{n-1}| < \varepsilon \)

\[
\frac{|p_n - p_{n-1}|}{|p_n|} < \varepsilon, \quad p_n \neq 0 \quad \text{or} \quad |f(p_n)| < \varepsilon
\]

• None is perfect. Use a combination in real applications.
Convergence

• Theorem
Suppose function $f(x)$ is continuous on $[a, b]$, and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero $p$ of $f(x)$ with
\[
|p_n - p| = \left(\frac{1}{2}\right)^n (b - a), \quad \text{when } n \geq 1
\]

• Convergence rate
The sequence $\{p_n\}_{n=1}^{\infty}$ converges to $p$ with the rate of convergence $O\left((\frac{1}{2})^n\right)$:
\[
p_n = p + O\left((\frac{1}{2})^n\right)
\]