2.6 Zeros of Polynomials and Horner's Method

Zeros of Polynomials

- **Definition**: Degree of a Polynomial

A **polynomial of degree $\boldsymbol{n}$** has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, a_n \neq 0$$

Where the $a_i' s$ are constants (either real or complex) called the **coefficients** of $P(x)$

- **Fundamental Theorem of Algebra**

If $P(x)$ is a polynomial of degree $n \geq 1$, with real or complex coefficients, $P(x) = 0$ has at least one root.

- **Corollary**

There exists unique constants $x_1, x_2, \ldots, x_k$ and unique positive integers $m_1, m_2, \ldots, m_k$ such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \ldots (x - x_k)^{m_k}$$

Remark:

1. Collection of zeros is unique
2. m_i are multiplicities of the individual zeros
3. A polynomial of degree n has exactly n zeros, counting multiplicity.

- **Corollary**

Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$ with $k > n$ are distinct numbers with $P(x_i) = Q(x_i)$ for all $i = 1,2, \ldots, k$, then $P(x) = Q(x)$ for all values of x.

Remark:

If two polynomials of degree n agree at at least (n+1) points, then they must be the same.

Horner's Method

- Horner's method is a technique to evaluate polynomials quickly. Need n multiplications and n additions to evaluate $P(x_0)$.
- Assume $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Evaluate $P(x_0)$.

Let $b_n = a_n, b_k = a_k + b_{k+1} x_0,$

$$\text{for } k = (n-1), (n-2), \dots, 1, 0$$

Then $b_0 = P(x_0)$.

- At the same time, we also computed the decomposition:

$$P(x) = (x - x_0) Q(x) + b_0,$$

Where $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$

Remark:

1. $P'(x_0) = Q(x_0)$, which can be computed by using Horner’s method in (n-1) multiplications and (n-1) additions.
2. Horner’s method is nested arithmetic

$$
\begin{aligned}
P(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\
&= (a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1) x + a_0 \\
&= ((a_n x^{n-2} + a_{n-1} x^{n-3} + \cdots) x + a_1) x + a_0 \\
&= \underbrace{(\ldots((}_{n-1} \underbrace{a_n x + a_{n-1}}_{b_{n-1}}) x + \cdots) x + a_1) x + a_0
\end{aligned}
$$