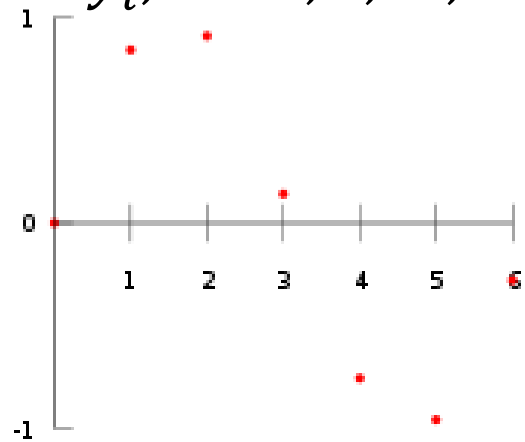
3.1 Interpolation and Lagrange Polynomial

Interpolation

- **Problem to be solved:** Given a set of $n+1$sample values of an unknown function f, we wish to determine a polynomial of degree n so that

$$P(x_i) = f(x_i) = y_i, i = 0, 1, \dots, n$$



$\boldsymbol{x}$	$\boldsymbol{f(x)}$
0	0
1	0.84
2	0.91
3	0.14
4	-0.76
…	…

Weierstrass Approximation theorem

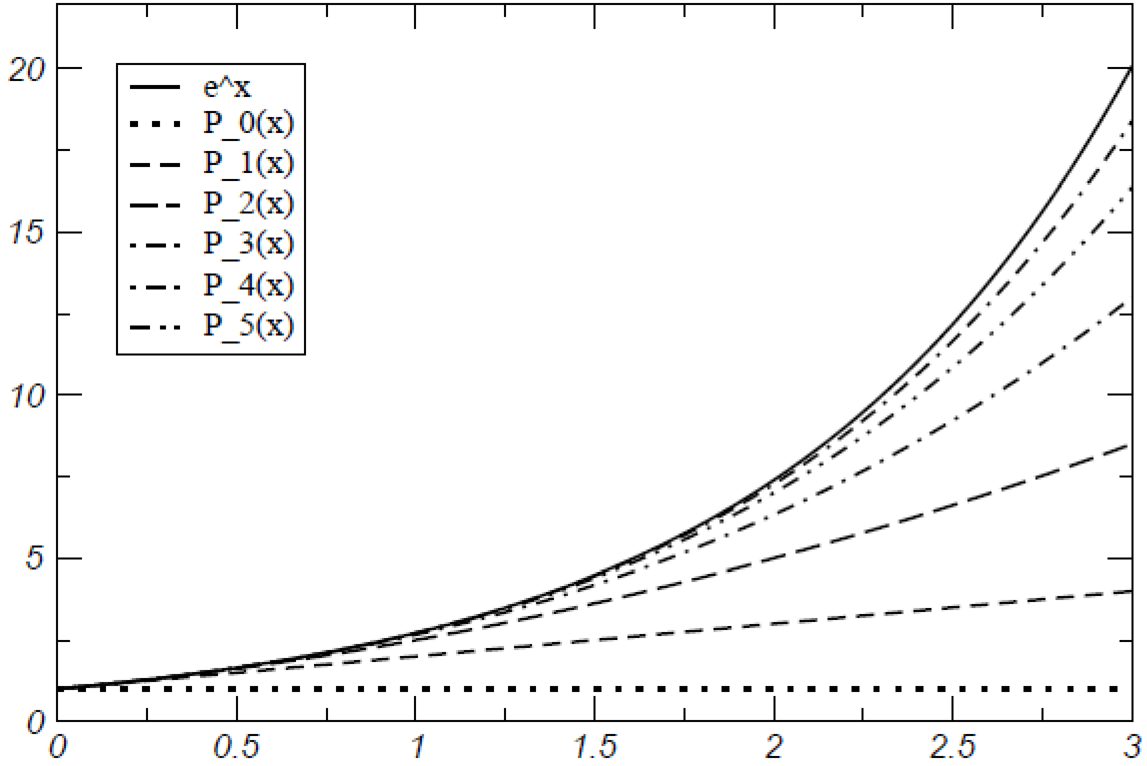
Suppose $f \in C[a,b]$. Then $\forall \epsilon > 0$, $\exists$ a polynomial $P(x)$:
$|f(x) - P(x)| < \epsilon, \forall x \in [a,b]$.

Remark:

1. The bound is uniform, i.e. valid for all x in $[a,b]$
2. The way to find $P(x)$ is unknown.

- **Question:** Can Taylor polynomial be used here?
- Taylor expansion is accurate in the neighborhood of **one** point. We need to the (interpolating) polynomial to pass many points.

- **Example**. Taylor approximation of e^x for $x \in [0,3]$



Linear Lagrange Interpolating Polynomial Passing through 2 Points

- **Problem:** Construct a functions passing through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

  First, define $L_0(x) = \frac{x-x_1}{x_0-x_1}, L_1(x) = \frac{x-x_0}{x_1-x_0}$

  Note: $L_0(x_0) = 1;\ \ L_0(x_1) = 0$

  $L_1(x_0) = 0;\ \ L_0(x_1) = 1$

  Then define the interpolating polynomial

  $$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

  Note:$P(x_0) = f(x_0),$ and $P(x_1) = f(x_1)$

  Claim: $P(x)$ is **the unique linear polynomial passing** through $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

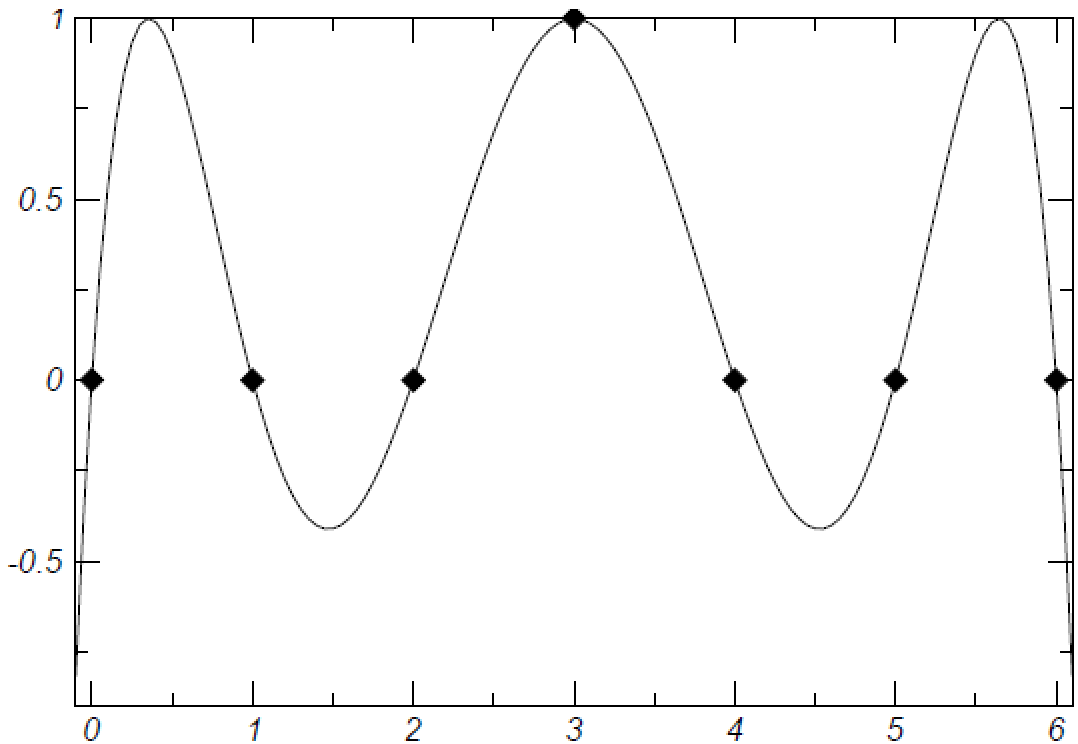
n-degree Polynomial Passing through $n+1$ Points

- Constructing a polynomial passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2)), \dots, (x_n, f(x_n))$.

  Define Lagrange basis functions

  $L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i} =$
  $\frac{x - x_0}{x_k - x_0} \dots \frac{x - x_{k-1}}{x_k - x_{k-1}} \cdot \frac{x - x_{k+1}}{x_k - x_{k+1}} \dots \frac{x - x_n}{x_k - x_n}$ for $k = 0, 1 \dots n$.

  Remark: $L_{n,k}(x_k) = 1; L_{n,k}(x_i) = 0, \ \forall i \neq k$.



- $L_{6,3}(x)$ for points $x_i = i,\ i = 0, \dots, 6$.

- **Theorem**. If $x_0, \ldots, x_n$ are $n+1$distinct numbers and f is a function whose values are given at these numbers, then a **unique polynomial** $P(x)$ of **degree at most $\boldsymbol{n}$** exists with $P(x_k) = f(x_k), \text{for each } k = 0, 1, \ldots n$.

$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x)$.

Where $L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$.

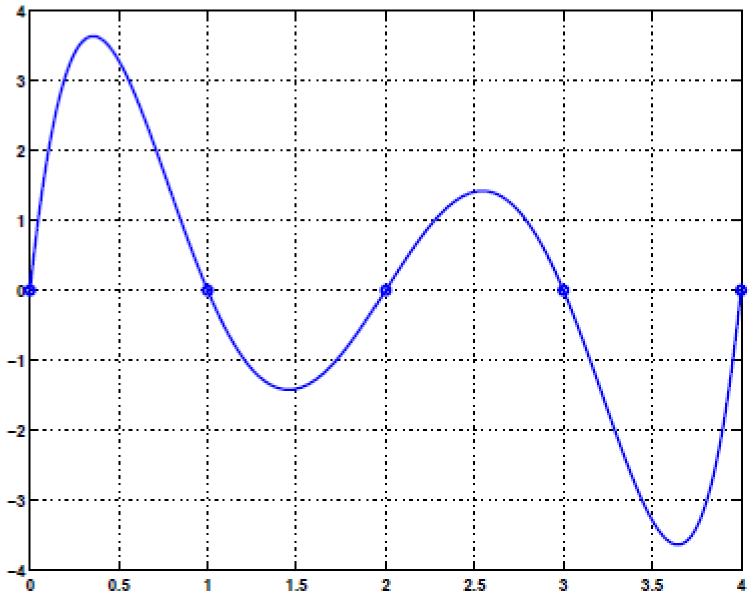
Error Bound for the Lagrange Interpolating Polynomial

Theorem. Suppose $x_0, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n).$$

Where $P(x)$ is the Lagrange interpolating polynomial.

- Remark:
  1. Applying the error term may be difficult. $(x - x_0)(x - x_1) \ldots (x - x_n)$ is oscillatory. $\xi(x)$ is generally unknown.
  2. The error formula is important as they are used for numerical differentiation and integration.



Plot of $(x - 0)(x - 1)(x - 2)(x - 3)(x - 4)$

Example. Suppose a table is to be prepared for $f(x) = e^x, x \in [0,1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0,1]$.