3.1 Interpolation and Lagrange Polynomial
Interpolation

- **Problem to be solved**: Given a set of $n + 1$ sample values of an unknown function $f$, we wish to determine a polynomial of degree $n$ so that

$$P(x_i) = f(x_i) = y_i, i = 0, 1, \ldots, n$$

**Weierstrass Approximation theorem**

Suppose $f \in C[a, b]$. Then $\forall \varepsilon > 0$, $\exists$ a polynomial $P(x)$: $$|f(x) - P(x)| < \varepsilon, \forall x \in [a, b].$$

**Remark:**
1. The bound is uniform, i.e. valid for all $x$ in $[a, b]$
2. The way to find $P(x)$ is unknown.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.84</td>
</tr>
<tr>
<td>2</td>
<td>0.91</td>
</tr>
<tr>
<td>3</td>
<td>0.14</td>
</tr>
<tr>
<td>4</td>
<td>-0.76</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
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</tbody>
</table>
• **Question:** Can Taylor polynomial be used here?

• Taylor expansion is accurate in the neighborhood of **one** point. We need to the (interpolating) polynomial to pass many points.

• **Example.** Taylor approximation of $e^x$ for $x \in [0,3]$
Linear Lagrange Interpolating Polynomial Passing through 2 Points

• **Problem:** Construct a function passing through two points \((x_0, f(x_0))\) and \((x_1, f(x_1))\).

First, define \(L_0(x) = \frac{x-x_1}{x_0-x_1}, L_1(x) = \frac{x-x_0}{x_1-x_0}\).

Note: \(L_0(x_0) = 1; \ L_0(x_1) = 0\)
\[L_1(x_0) = 0; \ L_0(x_1) = 1\]

Then define the interpolating polynomial

\[P(x) = L_0(x)f(x_0) + L_1(x)f(x_1)\]

Note: \(P(x_0) = f(x_0), \text{ and } P(x_1) = f(x_1)\)

Claim: \(P(x)\) is **the unique linear polynomial passing through \((x_0, f(x_0))\) and \((x_1, f(x_1))\)**.
$n$-degree Polynomial Passing through $n + 1$ Points

- Constructing a polynomial passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, ..., $(x_n, f(x_n))$.

Define Lagrange basis functions

$$L_{n,k}(x) = \prod_{i=0,i\neq k}^{n} \frac{x-x_i}{x_k-x_i} = \frac{x-x_0}{x_k-x_0} \cdot \frac{x-x_{k-1}}{x_k-x_{k-1}} \cdot \frac{x-x_{k+1}}{x_k-x_{k+1}} \cdot \frac{x-x_n}{x_k-x_n} \text{ for } k = 0,1 \ldots n.$$ 

Remark: $L_{n,k}(x_k) = 1; L_{n,k}(x_i) = 0, \forall i \neq k.$
• $L_{6,3}(x)$ for points $x_i = i$, $i = 0, \ldots, 6$. 
Theorem. If $x_0, \ldots, x_n$ are $n + 1$ distinct numbers and $f$ is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most $n$ exists with $P(x_k) = f(x_k)$, for each $k = 0, 1, \ldots, n$.

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x).$$

Where $L_{n,k}(x) = \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}$. 
Error Bound for the Lagrange Interpolating Polynomial

**Theorem.** Suppose \( x_0, \ldots, x_n \) are distinct numbers in the interval \([a, b]\) and \( f \in C^{n+1}[a, b] \). Then for each \( x \) in \([a, b]\), a number \( \xi(x) \) (generally unknown) between \( x_0, \ldots, x_n \), and hence in \((a, b)\), exists with

\[
f(x) = P(x) + \frac{f^{(n+)}(\xi(x))}{(n+1)!} (x - x_0)(x - x_1) \ldots (x - x_n).
\]

Where \( P(x) \) is the Lagrange interpolating polynomial.
• Remark:

1. Applying the error term may be difficult.
   
   \[(x - x_0)(x - x_1) \ldots (x - x_n)\] is oscillatory. \(\xi(x)\) is generally unknown.

2. The error formula is important as they are used for numerical differentiation and integration.

Plot of \((x - 0)(x - 1)(x - 2)(x - 3)(x - 4)\)
**Example.** Suppose a table is to be prepared for \( f(x) = e^x, \ x \in [0,1] \). Assume the number of decimal places to be given per entry is \( d \geq 8 \) and that the difference between adjacent \( x \)-values, the step size is \( h \). What step size \( h \) will ensure that linear interpolation gives an absolute error of at most \( 10^{-6} \) for all \( x \) in \([0,1]\).