3.4 Hermite Interpolation
3.5 Cubic Spline Interpolation

Hermite Polynomial

Definition. Suppose $f \in C^1[a, b]$. Let $x_0, \dots, x_n$ be distinct numbers in $[a, b]$, the Hermite polynomial $P(x)$ approximating f is that:

1. $P(x_i) = f(x_i)$, for $i = 0, \dots, n$

2. $\frac{dP(x_i)}{dx} = \frac{df(x_i)}{dx}$, for $i = 0, \dots, n$

Remark: $P(x)$ and $f(x)$ agree not only function values but also 1st derivative values at x_i, $i = 0, \dots, n$.

Theorem. If $f \in C^1[a,b]$ and $x_0, \dots, x_n \in [a,b]$ distinct numbers, the Hermite polynomial is:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

Where

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}(x_j)] L_{n,j}^2(x)$$
$$\hat{H}_{n,j}(x) = (x - x_j) L_{n,j}^2(x)$$

Moreover, if $f \in C^{2n+2}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

for some $\xi(x) \in (a,b)$.

Remark:

1. $H_{2n+1}(x)$ is a polynomial of degree at most $2n+1$.
2. $L_{n,j}(x)$ is jth Lagrange basis polynomial of degree n.
3. $\frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$ is the error term.

3rd Degree Hermite Polynomial

- Given distinct x_0, x_1 and values of f and f' at these numbers.

$$H_3(x) = \left(1 + 2\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f(x_0)$$

$$+ (x - x_0)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f'(x_0)$$

$$+ \left(1 + 2\frac{x_1 - x}{x_1 - x_0}\right)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f(x_1)$$

$$+ (x - x_1)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f'(x_1)$$

Hermite Polynomial by Divided Differences

Suppose $x_0, \dots, x_n$ and f, f' are given at these numbers.
Define $z_0, \dots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \qquad \text{for } i = 0, \dots, n$$

Construct divided difference table, but use

$$f'(x_0), f'(x_1), \dots, f'(x_n)$$

to set the following undefined divided difference:

$$f[z_0, z_1], f[z_2, z_3], \dots, f[z_{2n}, z_{2n+1}].$$

The Hermite polynomial is

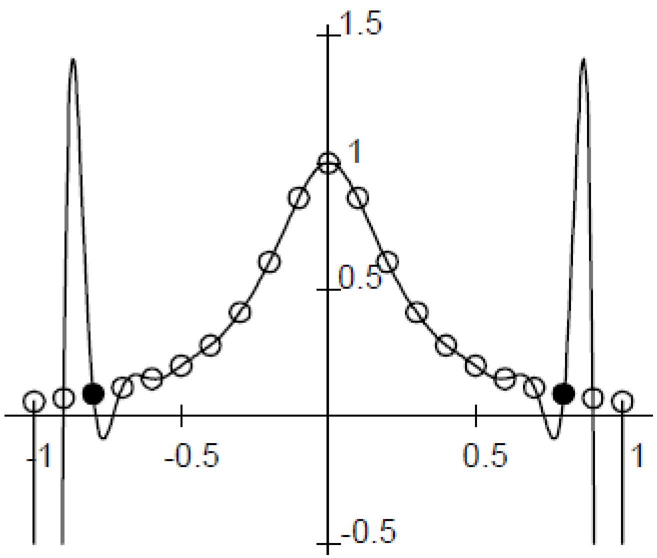
$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0) \dots (x - z_{k-1})$$

Divided Difference Notation for Hermite Interpolation

- Divided difference notation:

$$\begin{aligned} H_3(x) & \\ &= f(x_0) + f'(x_0)(x - x_0) \\ &+ f[x_0, x_0, x_1](x - x_0)^2 \\ &+ f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1) \end{aligned}$$

Problems with High Order Polynomial Interpolation



- 21 equal-spaced numbers to interpolate $f(x) = \frac{1}{1+16x^2}$. The interpolating polynomial oscillates between interpolation points.

Cubic Splines

- Idea: Use piecewise polynomial interpolation, i.e, divide the interval into smaller sub-intervals, and construct different low degree polynomial approximations (with small oscillations) on the sub-intervals.
- Challenge: If $f'(x_i)$ are not known, can we still generate interpolating polynomial with continuous derivatives?

Definition: Given a function f on $[a, b]$ and nodes $a = x_0 < \cdots < x_n = b$, a **cubic spline interpolant** S for f satisfies:

(a) $S(x)$ is a cubic polynomial $S_j(x)$ on $[x_j, x_{j+1}]$, $\forall j = 0,1, \dots, n-1$.

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$, $\forall j = 0,1, \dots, n-1$.

(c) $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$, $\forall j = 0,1, \dots, n-2$.

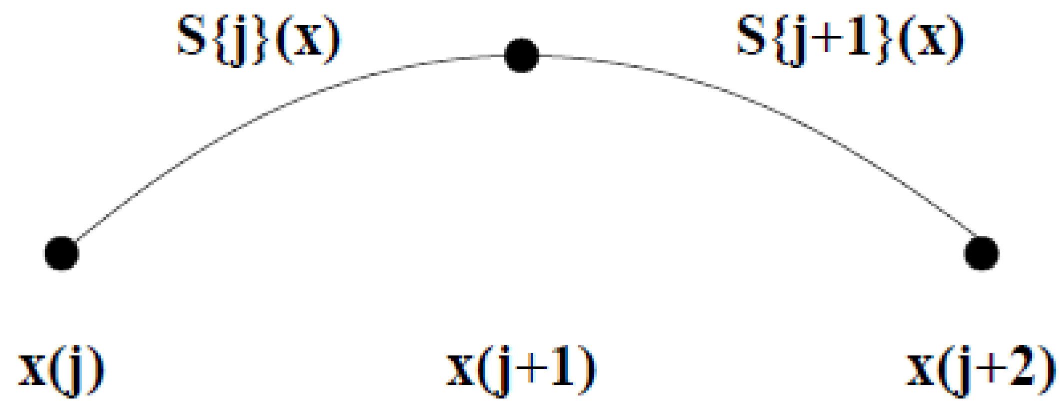
(d) $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$, $\forall j = 0,1, \dots, n-2$.

(e) $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$, $\forall j = 0,1, \dots, n-2$.

(f) One of the following boundary conditions:

(i) $S''\ \ (x_0) = S''\ \ (x_n) = 0$ (called free or natural boundary)

(ii) $S'\ \ (x_0) = f'(x_0)$ and $S'\ \ (x_n) = f'(x_n)$ (called clamped boundary)



Things to match at interior point x_{j+1}:

- The spline segment $S_j(x)$ is on $[x_j, x_{j+1}]$.
- The spline segment $S_{j+1}(x)$ is on $[x_{j+1}, x_{j+2}]$.
- Their function values: $S_j(x_{j+1}) = S_{j+1}(x_{j+1}) = f(x_{j+1})$
- First derivative values: $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$
- Second derivative values: $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$

Building Cubic Splines

- Define:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

and $h_j = x_{j+1} - x_j, \forall j = 0,1, \dots, n-1.$

Solve for coefficients a_j, b_j, c_j, d_j by:

1. $S_j(x_j) = a_j = f(x_j)$
2. $S_{j+1}(x_{j+1}) = a_{j+1} = a_j + b_j h_j + c_j (h_j)^2 + d_j (h_j)^3$
3. $S'_j(x_j) = b_j$, also $b_{j+1} = b_j + 2c_j h_j + 3d_j (h_j)^2$
4. $S''_j(x_j) = 2c_j$, also $c_{j+1} = c_j + 3d_j h_j$
5. Natural or clamped boundary conditions

Solving the Resulting Equations

$\forall j = 1, \dots, n-1$

$$h_{j-1}c_{j-1} + 2\left(h_{j-1} + h_j\right)c_j + h_j c_{j+1} = \frac{3}{h_j}\left(a_{j+1} - a_j\right) - \frac{3}{h_{j-1}}\left(a_j - a_{j-1}\right)$$

Remark: (n-1) equations for (n+1) unknowns $\{c_j\}_{j=0}^{n}$.

Once compute c_j, we then compute:

$$b_j = \frac{\left(a_{j+1} - a_j\right)}{h_j} - \frac{h_j\left(2c_j + c_{j+1}\right)}{3}$$

$$d_j = \frac{\left(c_{j+1} - c_j\right)}{3h_j}$$

Completing the System

- Natural boundary condition:
  1. $0 = S''_0(x_0) = 2c_0 \rightarrow c_0 = 0$
  2. $0 = S''_n(x_n) = 2c_n \rightarrow c_n = 0$
- Clamped boundary condition:
  a) $S'_0(x_0) = b_0 = f'(x_0)$
  b) $S'_{n-1}(x_n) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n) = f'(x_n)$

  Remark: a) and b) gives additional equations:

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(x_0)$$

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = -\frac{3}{h_0}(a_n - a_{n-1}) + 3f'(x_n)$$

Error Bound

If $f \in C^4[a,b]$, let $M = \max_{a \le x \le b} |f^4(x)|$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes: $a = x_0 < \cdots < x_n = b$, then with

$$h = \max_{0 \le j \le n-1} \left(x_{j+1} - x_j\right)$$

$$\max_{a \le x \le b} |f(x) - S(x)| \le \frac{5Mh^4}{384}.$$