4.6 Adaptive Quadrature Method

Motivation: 1) Given a tolerance, how do we make sure that approximation is within the accuracy (if we do not know much about integrand function)?

2) To reduce computational cost, we use less number of points when functional variation is small and more points when functional variation is large.

Main idea: Consider to use Simpson's rule to approximate $\int_a^b f(x)dx$ that is accurate to within ε.

Step 1) $\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi)$, with $S(a,b) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$, and $h = \frac{b-a}{2}$

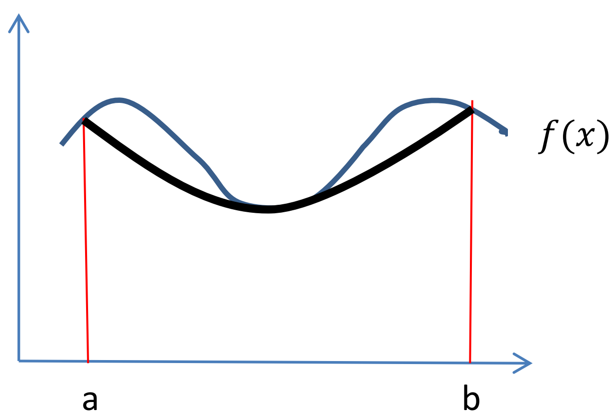

Figure 1: Step 1

Step 2) Use Composite Simpson's rule with $n = 4$ and corresponding step size $\frac{b-a}{4} = \frac{h}{2}$.

Remark: **Step 2)** is equivalent to divide$[a, b]$ to two equal length subintervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ and apply Simpson's rule on each subintervals.

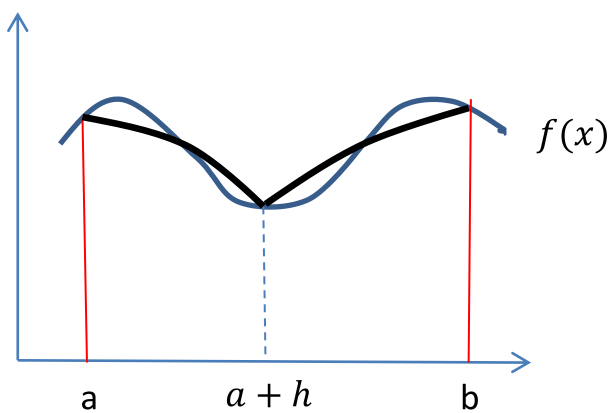

Figure 2: Step 2, Simpon's rule is used on two subintervals

$$\int_a^b f(x)dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \left(\frac{1}{16}\right)\frac{h^5}{90} f^{(4)}(\tilde{\xi}) \qquad \textbf{Eq. (1)}$$

With

$$S\left(a, \frac{a+b}{2}\right) = \frac{h}{6}\left[f(a) + 4f\left(a + \frac{h}{2}\right) + f(a+h)\right]$$

and

$$S\left(\frac{a+b}{2}, b\right) = \frac{h}{6}\left[f(a+h) + 4f\left(a + \frac{3h}{2}\right) + f(b)\right]$$

So, $S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \left(\frac{1}{16}\right)\frac{h^5}{90} f^{(4)}(\tilde{\xi}) \approx S(a, b) - \frac{h^5}{90} f^{(4)}(\xi)$.

Assume $\tilde{\xi} \approx \xi$.

$\frac{h^5}{90} f^{(4)}(\xi) \approx \frac{16}{15}\left[S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right]$. Substitute into **Eq. (1).**

$$\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{15}\left|S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right|$$

So if $\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| < 15\varepsilon$, then

$$\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| < \varepsilon$$

Step 3) If the result is NOT within the accuracy, use the same error estimate procedure to determine if the adaptive approximation to $\int_a^{\frac{a+b}{2}} f(x)dx$ and $\int_{\frac{a+b}{2}}^b f(x)dx$ is within the tolerance of $\frac{\varepsilon}{2}$. If NOT, keep dividing subintervals and repeat this procedure.