5.10 Stability (cont’d)

Example. Show modified Euler method \( w_{i+1} = w_i + \frac{h}{2} \left( f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i)) \right) \) is stable and convergent.

Solution \( \phi(t, w, h) = \frac{1}{2} f(t, w) + \frac{1}{2} f(t, w + hf(t, w)) \). Suppose \( f(t, w) \) satisfied a Lipschitz condition on \( \{(t, w) | a \leq t \leq b, \text{ and } -\infty < w < \infty \} \) with Lipschitz constant \( L \).

We next show that \( \phi(t, w, h) \) satisfies a Lipschitz condition in \( w \).

\[
|\phi(t, w_1, h) - \phi(t, w_2, h)| = \frac{1}{2} |f(t, w_1) + f(t, w_1 + h f(t, w_1)) - f(t, w_2) - f(t, w_2 + h f(t, w_2))| = \\
\frac{1}{2} |f(t, w_1) - f(t, w_2) + f(t, w_1 + h f(t, w_1)) - f(t, w_2 + h f(t, w_2))| \\
\leq \frac{1}{2} |f(t, w_1) - f(t, w_2)| + \frac{1}{2} |f(t, w_1 + h f(t, w_1)) - f(t, w_2 + h f(t, w_2))| \\
\leq \frac{1}{2} L|w_1 - w_2| + \frac{1}{2} L|w_1 + h f(t, w_1) - (w_2 + h f(t, w_2))| \\
\leq L|w_1 - w_2| + \frac{1}{2} L h|f(t, w_1) - f(t, w_2)| \\
\leq L|w_1 - w_2| + \frac{1}{2} h L^2 |w_1 - w_2| = |w_1 - w_2|(L + \frac{1}{2} h L^2)
\]

Therefore, \( \phi(t, w, h) \) satisfies a Lipschitz condition in \( w \) with constant \( (L + \frac{1}{2} h L^2) \) on \( \{(t, w, h) | a \leq t \leq b, -\infty < w < \infty, \text{and} \ h < h_0 \} \).

Also, if \( f(t, w) \) is continuous on \( \{(t, w) | a \leq t \leq b, -\infty < w < \infty \} \), then \( \phi(t, w, h) \) is continuous on \( \{(t, w, h) | a \leq t \leq b, -\infty < w < \infty, \text{and} \ h < h_0 \} \).

So, the modified Euler method is stable.

Moreover,

\[
\phi(t, w, 0) = \frac{1}{2} f(t, w) + \frac{1}{2} f(t, w) = f(t, w).
\]

This shows that the method is consistent, and the method is convergent.

The local truncation error of modified Euler method is \( O(h^2) \). So \( |y(t_i) - w_i| = O(h^2) \) by part (iii) of the theorem.

Definition. A \( m \)-step multistep is \textbf{consistent} if \( \lim_{h \to 0} |\tau_i(h)| = 0 \), for all \( i = m, m + 1, \ldots, N \) and \( \lim_{h \to 0} |\alpha_i - y(t_i)| = 0 \), for all \( i = 1, 2, \ldots, m = 1 \).
**Theorem.** Suppose the IVP $y' = f(t, y), a \leq t \leq b, y(a) = \alpha$ is approximated by an explicit Adams predictor-corrector method with an $m$-step Adams-Bashforth predictor equation

$$w_{i+1} = w_i + h[b_{m-1}f(t_i, w_i) + \cdots + b_0f(t_{i+1-m}, w_{i+1-m})]$$

with local truncation error $\tau_{i+1}(h)$ and an $(m-1)$-step implicit Adams-Moulton corrector equation

$$w_{i+1} = w_i + h[b_{m-1}f(t_i, w_i) + \cdots + b_0f(t_{i+2-m}, w_{i+2-m})]$$

with local truncation error $\tilde{\tau}_{i+1}(h)$. In addition, suppose that $f(t, y)$ and $f_y(t, y)$ are continuous on $\{(t, y) | a \leq t \leq b, \text{and} -\infty < y < \infty\}$ and that $f_y(t, y)$ is bounded. Then the local truncation error $\sigma_{i+1}(h)$ of the predictor-corrector method is

$$\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h)\tilde{b}_{m-1}f_y(t_{i+1}, \theta_{i+1})$$

where $\theta_{i+1}$ is a number between zero and $h\tau_{i+1}(h)$.

Moreover, there exist constant $k_1$ and $k_2$ such that

$$|w_i - y(t_i)| \leq \max_{0 \leq j \leq m-1} |w_j - y(t_j)| + k_1\sigma(h) e^{k_2(h_{i-a})}$$

where $\sigma(h) = \max_{m \leq j \leq N} |\sigma_j(h)|$.

**Example.** Consider the IVP $y' = 0, \quad 0 \leq t \leq 10, \quad y(0) = 1$, which is solved by $w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-2}))$. If in each step, there is a round-off error $\varepsilon$, and $w_1 = 1 + \varepsilon$. Find out how error propagates with respect to time.

**Solution:**

- $w_2 = -4(1 + \varepsilon) + 5(1) = 1 - 4\varepsilon$
- $w_3 = -4(1 - \varepsilon) + 5(1 + \varepsilon) = 1 + 21\varepsilon$
- $w_4 = -4(1 + 21\varepsilon) + 5(1 - 4\varepsilon) = 1 - 104\varepsilon$.

**Definition.** Consider to solve the IVP: $y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$. by an $m$-step multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m}$$

$$h[bmf(t_{i+1}, w_{i+1}) + b_{m-1}f(t_i, w_i) + \cdots + b_0f(t_{i+2-m}, w_{i+2-m})]$$

The **characteristic polynomial** of the method is given by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$ 

**Remark:**

1. The stability of a multistep method with respect to round-off error is dictated by magnitudes of zeros of the characteristic polynomial.
2. Solutions to $y' = 0, \quad y(0) = \alpha$ are expressed as $w_n = \alpha + \sum_{i=2}^{m} c_{i}\lambda_i^n$. 

2
**Definition.** Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0 = 0$$

associated with the $m$-step multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m}$$

$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})].$$

If $|\lambda_i| \leq 1$ and all roots with absolute value 1 are simple roots, then the difference equation is said to satisfy the root condition.

**Stability of multistep method**

**Definition of stability of multistep method.**

1) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called strongly stable.

2) Methods that satisfy the root condition and have more than one distinct roots with magnitude one are called weakly stable.

3) Methods that do not satisfy the root condition are called unstable.

**Example.** Show 4th order Adams-Bashforth method

$$w_{i+1} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

is strongly stable.

**Solution:** The characteristic equation of the 4th order Adams-Bashforth method is

$$P(\lambda) = \lambda^4 - \lambda^3 = 0$$

$$0 = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$. Therefore $P(\lambda)$ satisfies root condition and the method is strongly stable.

**Example.** Show 4th order Milne’s method
\[ w_{i+1} = w_{i-3} + \frac{4h}{3} [2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})] \]

is weakly stable.

**Solution:** The characteristic equation \( P(\lambda) = \lambda^4 - 1 = 0 \)

\[
0 = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1)
\]

\( P(\lambda) \) has roots \( \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i \).

All roots have magnitude one. So the method is weakly stable.

**Theorem.** A multistep method

\[
w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m} \\
+ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\
+ b_0 f(t_{i+1-m}, w_{i+1-m})],
\]

is stable if and only if it satisfies the root condition. If it is also consistent, then it is stable if and only if it is convergent.
5.11 Stiff Differential Equation

Example. The initial-value problem \( y' = -30y, \quad 0 \leq t \leq 1.5, \quad y(0) = \frac{1}{3} \) has exact solution \( y(t) = \frac{1}{3} e^{-30t} \). Use 4-stage Runge-Kutta method to solve with step size \( h = 0.1 \).

Facts:
1) A stiff differential equation is numerically unstable unless the step size is extremely small
2) Stiff differential equations are characterized as those whose exact solution has a term of the form \( e^{-ct} \), where \( c \) is a large positive constant.
3) Large derivatives of \( e^{-ct} \) give error terms that are dominating the solution.

Definition. The test equation is said to be \( y' = \lambda y, \quad y(0) = \alpha \), where \( \lambda < 0 \)
The test equation has exact solution \( y(t) = \alpha e^{\lambda t} \).

Euler’s Method for Test Equation

\[
\begin{align*}
  w_0 &= \alpha \\
  w_{j+1} &= w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)(1 + h\lambda)w_{j-1} = \cdots = (1 + h\lambda)^{j+1}\alpha & \text{for } j = 0, 1, \ldots, N - 1
\end{align*}
\]

The absolute error is \( |y(t_j) - w_j| = |e^{j\lambda} - (1 + h\lambda)^j||\alpha| = |(e^{h\lambda})^j - (1 + h\lambda)^j||\alpha| \)

So 1) the accuracy is determined by how well \((1 + h\lambda)\) approximate \(e^{h\lambda}\).

2) \((e^{h\lambda})^j\) decays to zero as \( j \) increases. \((1 + h\lambda)^j\) will decay to zero only if \(|1 + h\lambda| < 1\). This implies that \(-2 < h\lambda < 0\) or \( h < 2/|\lambda| \).

Note: Euler’s method is expected to be stable for the test equation only if \( h < 2/|\lambda| \).
Also, define \( Q(h\lambda) = 1 + h\lambda \) for Euler’s method, then \( w_{j+1} = Q(h\lambda)w_j \).
Now suppose a round-off error $\delta_0$ is introduced in the initial condition for Euler’s method
\[ w_0 = \alpha + \delta_0 \]
\[ w_j = (1 + h\lambda)^j(\alpha + \delta_0) \]
At the jth step, the round-off error is $\delta_j = (1 + h\lambda)^j\delta_0$.
So with $\lambda < 0$, the condition for control of the growth of round-off error is the same as the condition for controlling the absolute error $|1 + h\lambda| < 1$.

**Nth-order Taylor Method for Test Equation**
Applying the nth-order Taylor method to the test equation leads to
\[
\left| 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n \right| < 1
\]
to have stability. Also, define $Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n$ for a nth-order Taylor method, i.e., $w_{j+1} = Q(h\lambda)w_j$.

**Multistep Method for Test Equation**
Apply a multistep method to the test equation:
\[
w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m} + h\lambda[b_mw_{i+1} + b_{m-1}w_i + \cdots + b_0w_{i+1-m}],
\]
or
\[
(1 - h\lambda b_m)w_{i+1} - (a_{m-1} - h\lambda b_{m-1})w_i - \cdots - (a_0 - h\lambda b_0)w_{i+1-m} = 0
\]
Define the associated characteristic polynomial to this difference equation
\[
Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} - h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 - h\lambda b_0).
\]
Let $\beta_1, \beta_2, \ldots, \beta_m$ be the zeros of the characteristic polynomial to the difference equation.
Then $c_1, c_2, \ldots, c_m$ exist with
\[
w_i = \sum_{k=1}^{m} c_k(\beta_k)^i, \quad \text{for} \ i = 0, \ldots, N
\]
and $|\beta_k| < 1$ is required for stability.
Region of Stability

Definition. The region $R$ of absolute stability for a one-step method is $R = \{h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in \mathbb{C} \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda)\}$.

A numerical method is said to be A-stable if its region $R$ of absolute stability contains the entire left half-plane.

The only A-stable multistep method is implicit Trapezoidal method.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + \frac{h}{2} [f(t_j, w_j) + f(t_{j+1}, w_{j+1})], \quad \text{for } 0 \leq j \leq N - 1.$$  

The A-stable implicit backward Euler method.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + hf(t_{j+1}, w_{j+1}), \quad \text{for } 0 \leq j \leq N - 1.$$  

Remark: The technique commonly used for stiff systems is implicit methods.

Example. Show Backward Euler method has $Q(h\lambda) = \frac{1}{1-h\lambda}$.

Solution: $w_{j+1} = w_j + h\lambda w_{j+1}$

$$w_{j+1} = \frac{1}{1-h\lambda} w_j = \left(\frac{1}{1-h\lambda}\right)^2 w_{j-1} = \cdots = \left(\frac{1}{1-h\lambda}\right)^{j+1} w_0$$

Stability implies $\left|\frac{1}{1-h\lambda}\right| < 1$. 