5.10 Stability (cont'd)

Example. Show modified Euler method $w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right)$ is stable and convergent.

Solution $\phi(t, w, h) = \frac{1}{2}f(t, w) + \frac{1}{2}f(t, w + hf(t, w))$. Suppose $f(t, w)$ satisfied a Lipschitz condition on $\{(t, w)\,|a \le t \le b,$ and $-\infty < w < \infty\}$ with Lipschitz constant L.

We next show that $\phi(t, w, h)$ satisfies a Lipschitz condition in w.

$$|\phi(t, w_1, h) - \phi(t, w_2, h)| = \frac{1}{2}\left|f(t, w_1) + f\left(t, w_1 + hf(t, w_1)\right) - f(t, w_2) - f\left(t, w_2 + hf(t, w_2)\right)\right| =$$

$$\frac{1}{2}\left|f(t, w_1) - f(t, w_2) + f\left(t, w_1 + hf(t, w_1)\right) - f\left(t, w_2 + hf(t, w_2)\right)\right|$$

$$\le \frac{1}{2}|f(t, w_1) - f(t, w_2)| + \frac{1}{2}\left|f\left(t, w_1 + hf(t, w_1)\right) - f\left(t, w_2 + hf(t, w_2)\right)\right|$$

$$\le \frac{1}{2}L|w_1 - w_2| + \frac{1}{2}L\left|w_1 + hf(t, w_1) - \left(w_2 + hf(t, w_2)\right)\right| \le L|w_1 - w_2| + \frac{1}{2}Lh|f(t, w_1) - f(t, w_2)|$$

$$\le L|w_1 - w_2| + \frac{1}{2}hL^2|w_1 - w_2| = |w_1 - w_2|(L + \frac{1}{2}hL^2)$$

Therefore, $\phi(t, w, h)$ satisfies a Lipschitz condition in w with constant $(L + \frac{1}{2}hL^2)$ on $\{(t, w, h)\,|a \le t \le b, -\infty < w < \infty, and\ h < \ h_0\}$.

Also, if $f(t, w)$ is continuous on $\{(t, w)\,|a \le t \le b,$ and $-\infty < w < \infty\}$, then $\phi(t, w, h)$ is continuous on $\{(t, w, h)\,|a \le t \le b, -\infty < w < \infty, and\ h < \ h_0\}$.

So, the modified Euler method is stable.

Moreover,

$$\phi(t, w, 0) = \frac{1}{2}f(t, w) + \frac{1}{2}f(t, w + 0f(t, w)) = f(t, w).$$

This shows that the method is consistent, and the method is convergent.

The local truncation error of modified Euler method is $O(h^2)$. So $|y(t_i) - w_i| = O(h^2)$ by part (iii) of the theorem.

Definition. A m-step multistep is **consistent** if $\lim_{h\to 0}|\tau_i(h)| = 0$, for all $i = m, m+1, \dots, N$ and $\lim_{h\to 0}|\alpha_i - y(t_i)| = 0$, for all $i = 1, 2, \dots, m = 1$.

Theorem. Suppose the IVP $y' = f(t,y), a \le t \le b,\ y(a) = \alpha$ is approximated by an explicit Adams predictor-corrector method with an m-step Adams-Bashforth predictor equation $w_{i+1} = w_i + h[b_{m-1}f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})]$ with local truncation error $\tau_{i+1}(h)$ and an (m-1)-step implicit Adams-Moulton corrector equation $w_{i+1} = w_i + h[\tilde{b}_{m-1}f(t_i, w_i) + \cdots + \tilde{b}_0 f(t_{i+2-m}, w_{i+2-m})]$ with local truncation error $\tilde{\tau}_{i+1}(h)$. In addition, suppose that $f(t,y)$ and $f_y(t,y)$ are continuous on $\{(t,y)\ |a \le t \le b, \text{and} -\infty < y < \infty\}$ and that $f_y(t,y)$ is bounded. Then the local truncation error $\sigma_{i+1}(h)$ of the predictor-corrector method is

$$\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h)\tilde{b}_{m-1}f_y(t_{i+1}, \theta_{i+1})$$

where θ_{i+1} is a number between zero and $h\tau_{i+1}(h)$.
Moreover, there exist constant k_1 and k_2 such that

$$|w_i - y(t_i)| \le \left[\max_{0 \le j \le m-1} \left|w_j - y(t_j)\right| + k_1\sigma(h)\right] e^{k_2(t_i - a)}$$

where $\sigma(h) = \max_{m \le j \le N} \left|\sigma_j(h)\right|$.

Example. Consider the IVP $y' = 0, \quad 0 \le t \le 10,\ y(0) = 1,$ which is solved by $w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-2}))$. If in each step, there is a round-off error ε, and $w_1 = 1 + \varepsilon$. Find out how error propagates with respect to time.

Solution: $w_2 = -4(1+\varepsilon) + 5(1) = 1 - 4\varepsilon$

$$w_3 = -4(1-\varepsilon) + 5(1+\varepsilon) = 1 + 21\varepsilon$$
$$w_4 = -4(1+21\varepsilon) + 5(1-4\varepsilon) = 1 - 104\varepsilon.$$

Definition. Consider to solve the IVP: $y' = f(t,y), \quad a \le t \le b,\ y(a) = \alpha$. by an m-step multistep method

$$\begin{gathered} w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1}f(t_i, w_i) + \cdots \\ + b_0 f(t_{i+1-m}, w_{i+1-m})], \end{gathered}$$

The **characteristic polynomial** of the method is given by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

Remark: (1) The stability of a multistep method with respect to round-off error is dictated by magnitudes of zeros of the characteristic polynomial. (2) Solutions to $y' = 0, \quad y(0) = \alpha$ are expressed as $w_n = \alpha + \sum_{i=2}^{m} c_i \lambda_i^n$.

Definition. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the roots of the **characteristic equation**

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0 = 0$$

associated with the *m*-step multistep method

$$\begin{gathered} w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\ +b_0 f(t_{i+1-m}, w_{i+1-m})], \end{gathered}$$

If $|\lambda_i| \le 1$ and all roots with absolute value 1 are simple roots, then the difference equation is said to satisfy the **root condition**.

Stability of multistep method

Definition of stability of multistep method.

1) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly stable**.
2) Methods that satisfy the root condition and have more than one distinct roots with magnitude one are called **weakly stable**.
3) Methods that do not satisfy the root condition are called **unstable**.

Example. Show 4th order Adams-Bashforth method

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

is strongly stable.

Solution: The characteristic equation of the 4th order Adams-Bashforth method is

$$P(\lambda) = \lambda^4 - \lambda^3 = 0$$

$$0 = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$.

Therefore $P(\lambda)$ satisfies root condition and the method is strongly stable.

Example. Show 4th order Miline's method

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})]$$

is weakly stable.

Solution: The characteristic equation $P(\lambda) = \lambda^4 - 1 = 0$

$$0 = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$.

All roots have magnitude one. So the method is weakly stable.

Theorem. A multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

is stable **if and only if** it satisfies the root condition. If it is also consistent, then it is stable **if and only if** it is convergent.

5.11 Stiff Differential Equation

Example. The initial-value problem $y' = -30y, \quad 0 \le t \le 1.5, \ y(0) = \frac{1}{3}$ has exact solution $y(t) = \frac{1}{3}e^{-30t}$. Use 4-stage Runge-Kutta method to solve with step size $h = 0.1$.

Facts:

1) A stiff differential equation is numerically unstable unless the step size is extremely small

2) Stiff differential equations are characterized as those whose exact solution has a term of the form e^{-ct}, where c is a large positive constant.

3) Large derivatives of e^{-ct} give error terms that are dominating the solution.

Definition. The *test equation* is said to be $y' = \lambda y, \qquad y(0) = \alpha$, where $\lambda < 0$

The test equation has exact solution $y(t) = \alpha e^{\lambda t}$.

Euler's Method for Test Equation

$$w_0 = \alpha$$

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)(1 + h\lambda)w_{j-1} = \cdots = (1 + h\lambda)^{j+1}\alpha \qquad \text{for } j = 0,1,\dots,N-1$$

The absolute error is $|y(t_j) - w_j| = |e^{jh\lambda} - (1 + h\lambda)^j||\alpha| = |(e^{h\lambda})^j - (1 + h\lambda)^j||\alpha|$

So 1) the accuracy is determined by how well $(1 + h\lambda)$ approximate $e^{h\lambda}$.

2) $(e^{h\lambda})^j$ decays to zero as j increases. $(1 + h\lambda)^j$ will decay to zero only if $|1 + h\lambda| < 1$. This implies that $-2 < h\lambda < 0$ or $h < 2/|\lambda|$.

Note: Euler's method is expected to be stable for the test equation only if the step size $h < 2/|\lambda|$.

Also, **define** $Q(h\lambda) = 1 + h\lambda$ for Euler's method, then $w_{j+1} = Q(h\lambda)w_j$

Now suppose a round-off error δ_0 is introduced in the initial condition for Euler's method

$$w_0 = \alpha + \delta_0$$
$$w_j = (1 + h\lambda)^j(\alpha + \delta_0)$$

At the jth step, the round-off error is $\delta_j = (1 + h\lambda)^j\delta_0$.

So with $\lambda < 0$, the condition for control of the growth of round-off error is the same as the condition for controlling the absolute error $|1 + h\lambda| < 1$.

Nth-order Taylor Method for Test Equation

Applying the nth-order Taylor method to the test equation leads to

$$\left|1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n\right| < 1$$

to have stability. Also, define $Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n$ for a nth-order Taylor method, i.e., $w_{j+1} = Q(h\lambda)w_j$.

Multistep Method for Test Equation

Apply a multistep method to the test equation:

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h\lambda[b_m w_{i+1} + b_{m-1}w_i + \cdots$$
$$+b_0 w_{i+1-m}],$$

or

$$(1 - h\lambda b_m)w_{i+1} - (a_{m-1} - h\lambda b_{m-1})w_i - \cdots - (a_0 - h\lambda b_0)w_{i+1-m} = 0$$

Define the associated **characteristic polynomial** to this difference equation

$$Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} - h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 - h\lambda b_0).$$

Let $\beta_1, \beta_2, \dots, \beta_m$ be the zeros of the **characteristic polynomial** to the difference equation.

Then $c_1, c_2, \dots, c_m$ exist with

$$w_i = \sum_{k=1}^{m} c_k(\beta_k)^i, \qquad \text{for} \quad i = 0, \dots N$$

and $|\beta_k| < 1$ is required for stability.

Region of Stability

Definition. The **region *R* of absolute stability** for a one-step method is $R = \{h\lambda \in C \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in C \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda)\}$.

A numerical method is said to be A-stable if its region R of absolute stability contains the entire left half-plane.

The only A-stable multistep method is **implicit Trapezoidal method**.

$$w_0 = \alpha$$
$$w_{j+1} = w_j + \frac{h}{2}\left[f(t_j, w_j) + f(t_{j+1}, w_{j+1})\right], \qquad \text{for} \quad 0 \le j \le N-1.$$

The A-stable **implicit backward Euler method**.

$$w_0 = \alpha$$
$$w_{j+1} = w_j + hf(t_{j+1}, w_{j+1}), \qquad \text{for} \quad 0 \le j \le N-1.$$

Remark: The technique commonly used for stiff systems is implicit methods.

Example. Show Backward Euler method has $Q(h\lambda) = \frac{1}{1-h\lambda}$.

Solution: $w_{j+1} = w_j + h\lambda w_{j+1}$

$$w_{j+1} = \frac{1}{1-h\lambda} w_j = \left(\frac{1}{1-h\lambda}\right)^2 w_{j-1} = \cdots = \left(\frac{1}{1-h\lambda}\right)^{j+1} w_0$$

Stability implies $\left|\frac{1}{1-h\lambda}\right| < 1$.