5.3 High-Order Taylor Methods

Consider the IVP

$$y' = f(t,y), \quad a \le t \le b, \; y(a) = \beta.$$

Definition: The difference method

$$w_0 = \beta$$

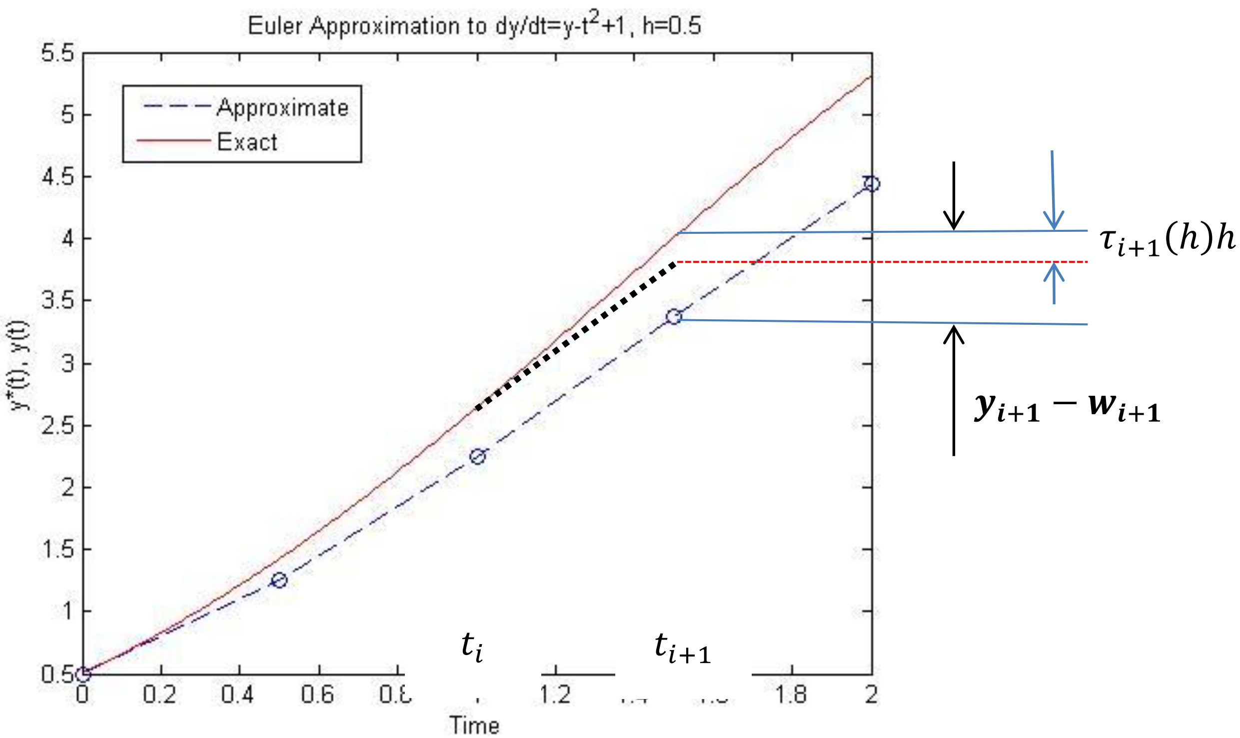
$$w_{i+1} = w_i + h\phi(t_i, w_i), \qquad \text{for each } i = 0,1,2,\cdots,N-1, \quad \text{with step size } h = \frac{b-a}{N}$$

has **Local Truncation Error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - \left(y_i + h\phi(t_i,y_i)\right)}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i,y_i) \qquad \text{for each } i = 0,1,2,\cdots,N-1.$$

Note: $y_i \coloneqq y(t_i)$ and $y_{i+1} \coloneqq y(t_{i+1})$.

<u>Geometric view of local truncation error</u>



Example. Analyze the local truncation error of Euler's method for solving $y' = f(t,y), \quad a \le t \le b, \;\; y(a) = \beta$. Assume $|y''(t)| < M$ with $M > 0$ constant.

Solution:

$$\tau_{i+1}(h) = \frac{y_{i+1} - \left(y_i + hf(t_i, y_i)\right)}{h} = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) = \frac{y(t_i) + hf(t_i, y_i) + \frac{h^2}{2} y''(\xi_i) - y_i}{h} - f(t_i, y_i)$$

with $\xi_i \in (t_i, t_{i+1})$.

$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i)$.

Thus $|\tau_{i+1}(h)| \le \frac{h}{2} M$.

So the local truncation error in Euler's method is $O(h)$.

Consider the IVP

$$y' = f(t,y), \quad a \le t \le b, \;\; y(a) = \beta.$$

Compute $y'', y^{(3)} \cdots$.

First, by IVP: $y'' = f'(t, y(t))$

$$y^{(3)}(t) = f''(t, y(t))$$
$$\vdots$$
$$y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Second, by chain rule:

$$y'' (= \frac{dy'(t)}{dt}) = \frac{df(t, y(t))}{dt} = \frac{\partial f}{\partial t}\big(t, y(t)\big) + \frac{\partial f}{\partial y}\big(t, y(t)\big) \cdot y'(t) = \frac{\partial f}{\partial t}\big(t, y(t)\big) + \frac{\partial f}{\partial y}\big(t, y(t)\big) \cdot f(t, y(t))$$
$$\vdots$$

<u>Derivation of higher-order Taylor methods</u>

Consider the IVP

$$y' = f(t,y), \quad a \le t \le b, \;\; y(a) = \beta, \qquad \text{with step size } h = \frac{b-a}{N}, \quad t_{i+1} = a + ih.$$

Expand $y(t)$ in the *n*th Taylor polynomial about t_i, evaluated at t_{i+1}

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$
$$= y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2}f'\big(t_i, y(t_i)\big) + \cdots + \frac{h^n}{n!}f^{(n-1)}\big(t_i, y(t_i)\big) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))$$

for some $\xi_i \in (t_i, t_{i+1})$. Delete remainder term to obtain the *n*th Taylor method of order *n*.

Denote $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i)$

<u>Taylor method of order *n*</u>

$w_0 = \beta$

$w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$ for each $i = 0, 1, 2, \cdots, N-1$.

Remark: Euler's method is Taylor method of order one.

Example 1. Use Taylor method of orders (a) two and (b) four with N = 10 to the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution:

$h = \frac{2-0}{N} = \frac{2-0}{10} = 0.2$. So $t_i = 0 + 0.2i = 0.2i$ for each $i = 0, 1, 2, \cdots, 10$.

(a) $f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t$

So $T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) = (w_i - t_i^2 + 1) + 0.1\big(w_i - t_i^2 + 1 - 2t_i\big) = 1.1\big(w_i - t_i^2 + 1\big) - 0.2t_i$

The 2nd order Taylor method is

$$w_0 = 0.5$$
$$w_{i+1} = w_i + 0.2\big(1.1\big(w_i - t_i^2 + 1\big) - 0.2t_i\big) \qquad \text{for each } i = 0, 1, 2, \cdots, 9$$

Now compute approximations at each time step:

$$w_0 = 0.5$$
$$w_1 = w_0 + 0.2\left(1.1(w_0 - (0)^2 + 1) - 0.2(0)\right) = 0.83; \qquad \text{abs. eror:} \quad |w_1 - y_1| = 0.000701$$
$$w_2 = w_2 + 0.2(1.1(w_2 - (0.2)^2 + 1) - 0.2(0.2)) = 1.2158; \qquad \text{abs. eror:} \quad |w_2 - y_2| = 0.001712$$
$$\vdots$$

(b) $f''(t, y(t)) = \frac{d}{dt}(f') = (y - t^2 + 1 - 2t)' = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$

$$f^{(3)}(t, y(t)) = \frac{d}{dt}(f'') = (y - t^2 - 2t - 1)' = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$$

So $$T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f^{(3)}(t_i, w_i)$$

$$= (w_i - t_i^2 + 1) + \frac{h}{2}\left(w_i - t_i^2 + 1 - 2t_i\right) + \frac{h^2}{6}\left(w_i - t_i^2 - 2t_i - 1\right) + \frac{h^3}{24}\left(w_i - t_i^2 - 2t_i - 1\right)$$

$$= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)\left(w_i - t_i^2\right) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}$$

The 4th order Taylor method is

$$w_0 = 0.5$$

$$w_{i+1} = w_i + h\left(\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)\left(w_i - t_i^2\right) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\right)$$

for each $i = 0, 1, 2, \cdots, 9.$

Now compute approximate solutions at each time step:

$$w_1 = 0.5 + 0.2\left(\left(1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24}\right)(0.5 - 0) - \left(1 + \frac{0.2}{3} + \frac{0.2^2}{12}\right)(0) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24}\right) = 0.8293$$

$$\text{abs. eror of 4th order Taylor at } t_1: \quad |w_1 - y_1| = 0.000001$$

$$w_2 = 0.8293 + 0.2\left(\left(1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24}\right)(0.8293 - 0.2^2) - \left(1 + \frac{0.2}{3} + \frac{0.2^2}{12}\right)(0.2(0.2)) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24}\right)$$
$$= 1.214091$$

$$\text{abs. eror 4th order Taylor at } t_2: \quad |w_2 - y_2| = 0.000003$$

Finding approximations at time other than t_i

Example. (Table 5.4 on Page 259). Assume the IVP $y' = y - t^2 + 1,\quad 0 \le t \le 2,\quad y(0) = 0.5$ is solved by the 4th order Taylors method with time step size $h = 0.2$. $w_6 = 3.1799640$ ($t_6 = 1.2$), $w_7 = 3.7324321$ ($t_7 = 1.4$). Find $y(1.25)$.

Solution:

Method 1: use linear Lagrange interpolation.

$y(1.25) \approx \frac{1.25-1.4}{1.2-1.4} w_6 + \frac{1.25-1.2}{1.4-1.2} w_7 = 3.3180810$

Method 2: use Hermite polynomial interpolation (more accurate than the result by linear Lagrange interpolation).

First use $y' = y - t^2 + 1$ to approximate $y'(1.2)$ and $y'(1.4)$.

$y'(1.2) = y(1.2) - (1.2)^2 + 1 \approx 3.1799640 - (1.2)^2 + 1 = 2.7399640$

$y'(1.4) = y(1.4) - (1.4)^2 + 1 \approx 3.7324321 - (1.4)^2 + 1 = 2.7724321$

Then use **Theorem 3.9** to construct Hermite polynomial $H_3(x)$.

$y(1.25) \approx H_3(1.25)$.

Error analysis

Theorem 5.12 If Taylor method of order *n* is used to approximate the solution to the IVP

$$y' = f(t, y), \quad a \le t \le b,\ \ y(a) = \beta$$

with step size h and if $y \in C^{n+1}[a, b]$, then the **local truncation error** is $O(h^n)$.

Remark: $y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} f'\big(t_i, y(t_i)\big) + \cdots + \frac{h^n}{n!} f^{(n-1)}\big(t_i, y(t_i)\big) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}\big(\xi_i, y(\xi_i)\big).$$

$y^{(n+1)}(t) = f^{(n)}\big(t, y(t)\big)$ is bounded by $\left|y^{(n+1)}(t)\right| \le M$.

Thus $|\tau_{i+1}(h)| \le \frac{h^n}{(n+1)!} M$.

So the local truncation error in Euler's method is $O(h^n)$.

5.4 Runge-Kutta Methods

Motivation: Obtain high-order accuracy of Taylor's method without knowledge of derivatives of $f(t,y)$.

Theorem 5.13(Taylor's Theorem in Two Variables) Suppose $f(t,y)$ and partial derivative up to order $n+1$ continuous on $D=\{(t,y)|a\ \le t\le b, c\ \le y\le d\}$, let $(t_0,y_0)\in D$. For $(t,y)\in D$, there is $\xi\in[t,t_0]$ and $\mu\in[y,y_0]$ with

$$f(t,y)=P_n(t,y)+R_n(t,y).$$

Here

$$\begin{aligned}P_n(t,y)=f(t_0,y_0)&+\left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0)+(y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]\\&+\left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0)+(t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\partial y}(t_0,y_0)+\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right]\\&+\left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]\end{aligned}$$

$$R_n(t,y)=\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(t_0,y_0)$$

$P_n(t,y)$ is the nth Taylor polynomial in two variables.

<u>Derivation of Runge-Kutta method of order two</u>

1. Determine a_1, α_1, β_1 such that

$$a_1 f(t+\alpha_1,y+\beta_1)\approx f(t,y)+\frac{h}{2}f'(t,y)=T^{(2)}(t,y) \text{ with } O(h^2) \text{ error.}$$

Notice $f'(t,y)=\frac{df(t,y(t))}{dt}=\frac{\partial f}{\partial t}\big(t,y(t)\big)+\frac{\partial f}{\partial y}\big(t,y(t)\big)\cdot y'(t)=\frac{\partial f}{\partial t}\big(t,y(t)\big)+\frac{\partial f}{\partial y}\big(t,y(t)\big)\cdot f(t,y(t))$

We have $T^{(2)}(t,y)=f(t,y)+\frac{h}{2}\frac{\partial f}{\partial t}\big(t,y(t)\big)+\frac{h}{2}\frac{\partial f}{\partial y}\big(t,y(t)\big)\cdot f\big(t,y(t)\big)$ (1)

2. Expand $a_1 f(t+\alpha_1,y+\beta_1)$ in 1st degree Taylor polynomial:

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+\alpha_1, y+\beta_1) \quad (2)$$

3. Match coefficients of equation (1) and (2) gives

$$a_1 = 1, \qquad a_1\alpha_1 = \frac{h}{2}, \qquad a_1\beta_1 = \frac{h}{2} f\big(t, y(t)\big)$$

with unique solution

$$a_1 = 1, \qquad \alpha_1 = \frac{h}{2}, \qquad \beta_1 = \frac{h}{2} f\big(t, y(t)\big)$$

4. This gives

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t, y(t)\big)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t, y(t)\big)\right)$$

with $R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t, y(t)\big)\right) = O(h^2)$

<u>Midpoint Method (one of Runge-Kutta methods of order two)</u>

Consider the IVP

$$y' = f(t,y), \quad a \le t \le b, \ \ y(a) = \beta.$$

with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Local truncation error is $O(h^2)$.

<u>Two stage formula</u>:

$$w_0 = \beta$$

$$k_1 = f(t_i, w_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} k_1\right)$$

$$w_{i+1} = w_i + hk_2$$

Example 2. Use the Midpoint method with $N = 10, h = 0.2,\ t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$