5.4 Runge-Kutta Method (cont'd)

Modified Euler Method (Another Runge-Kutta methods of order two)

Consider the IVP

$$y' = f(t, y), \quad a \le t \le b, \ y(a) = \beta.$$

with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Local truncation error is $O(h^2)$.

Two stage formula:

$$w_0 = \beta$$
$$k_1 = f(t_i, w_i)$$
$$k_2 = f(t_{i+1}, w_i + hk_1)$$
$$w_{i+1} = w_i + \frac{h}{2}[k_1 + k_2]$$

Example. Use the Modified Euler method with $N = 10, h = 0.2, \ t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$

Heun's Method (Runge-Kutta Method of order three)

Idea: Approximate $T^3(t, y)$ with $O(h^3)$ error by $f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Runge-Kutta Method of order four

$$w_0 = \beta$$
$$k_1 = hf(t_i, w_i)$$
$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$$
$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$$
$$k_4 = hf(t_{i+1}, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$