5.6 Multistep Methods

**Motivation:** Solve the IVP: $y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha$. To compute solution at $t_{i+1}$, approximate solutions at mesh points $t_0, t_1, t_2, \ldots t_i$ are already obtained. Since in general error $|y(t_{i+1}) - w_{i+1}|$ grows with respect to time $t$, it then makes sense to use more previously computed approximate solution $w_i, w_{i-1}, w_{i-2}, \ldots$ when computing $w_{i+1}$.

**Adams-Bashforth two-step explicit method.**

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{2} [3 f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \quad \text{where } i = 1, 2, \ldots, N - 1.$$ 

**Adams-Moulton two-step implicit method.**

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{12} [5 f(t_{i+1}, w_{i+1}) + 8 f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \quad \text{where } i = 1, 2, \ldots, N - 1.$$ 

**Example.** Solve the IVP $y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$ by Adams-Bashforth two-step explicit method and Adams-Moulton two-step implicit method respectively. Use the exact values given by $y(t) = (t + 1)^2 - 0.5e^t$ to get needed starting values for approximation and $h = 0.2$.

**Solution:**

$$w_0 = 0.5$$

$$w_1 = y(0.2) = (0.2 + 1)^2 - 0.5e^{0.2} = 0.8292986 \quad \text{(by using } y(t))$$

1) Adams-Bashforth two-step explicit method

$$w_{i+1} = w_i + \frac{h}{2} [3 (w_i - t_i^2 + 1) - (w_{i-1} - t_{i-1}^2 + 1)]$$

$$w_2 = 0.8292986 + 0.1 [3(0.8292986 - 0.2^2 + 1) - (0.5 + 1)] = 1.2160882$$

$$w_3 = 1.2160882 + 0.1 [3(1.2160882 - 0.4^2 + 1) - (0.8292986 - 0.2^2 + 1)] = 1.6539848$$

.... and so on till to compute $w_{10}$. 


2) Adams-Moulton two-step implicit method

\[ w_{i+1} = w_i + \frac{h}{12} [5(w_{i+1} - t_{i+1}^2 + 1) + 8(w_i - t_i^2 + 1) - (w_{i-1} - t_{i-1}^2 + 1)] \]

\[ w_2 = 0.8292986 + \frac{0.2}{12} [5(w_2 - 0.4^2 + 1) + 8(0.8292986 - 0.2^2 + 1) - (0.5 + 1)] \]

Solve for \( w_2 \):

\[ w_2 = 1.21404191 \]

\[ w_3 = 1.21404191 + \frac{0.2}{12} [5(w_3 - 0.6^2 + 1) + 8(1.21404191 - 0.4^2 + 1) - (0.8292986 - 0.2^2 + 1)] \]

Solve for \( w_3 \): ...

... and so on till to compute \( w_{10} \).

**Definition**

An \( m \)-step multistep method for solving the IVP

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \]

has a difference equation for approximate \( w_{i+1} \) at \( t_{i+1} \):

\[ w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \ldots + a_0w_{i+1-m} \]

\[ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \ldots + b_0 f(t_{i+1-m}, w_{i+1-m})], \]

where \( h = \frac{b-a}{N} \), and starting values are specified:

\[ w_0 = \alpha, \quad w_1 = \alpha, \ldots, w_{m-1} = \alpha_{m-1}. \]

*Explicit* method if \( b_m = 0 \), *implicit* method if \( b_m \neq 0 \).
5.6 Multistep Methods (cont’d)

Example. Derive Adams-Bashforth two-step explicit method: Solve the IVP: \( y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha \).

Integrate \( y' = f(t,y) \) over \([y_i, y_{i+1}]\)

\[
y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} y'(t) \, dt = \int_{t_i}^{t_{i+1}} f(t, y(t)) \, dt
\]

Use \((t_i, y_i)\) and \((t_{i-1}, y_{i-1})\) to form interpolating polynomial \( P_1(t) \) (by Newton backward difference (Page 129)) to approximate \( f(t,y) \).

\[
\int_{t_i}^{t_{i+1}} f(t,y) \, dt = \int_{t_i}^{t_{i+1}} \left( f(t_i, y_i) + \nabla f(t_i, y_i) \frac{(t - t_i)}{h} + \text{error} \right) \, dt
\]

\[
y_{i+1} - y_i = h \left[ f(t_i, y_i) + \frac{1}{2} \left( f(t_i, y_i) - f(t_{i-1}, y_{i-1}) \right) \right] + \text{Error}
\]

where \( h = t_{i+1} - t_i \), and the backward difference \( \nabla f(t_i, y_i) = hf[t_i, t_{i-1}] = (f(t_i, y_i) - f(t_{i-1}, y_{i-1})) \).

Consequently, Adams-Bashforth two-step explicit method is:

\[
w_0 = \alpha, \quad w_1 = \alpha_1
\]

\[
w_{i+1} = w_i + \frac{h}{2} \left[ 3f(t_i, w_i) - f(t_{i-1}, w_{i-1}) \right] \quad \text{where } i = 1, 2, \ldots, N - 1.
\]

Local Truncation Error. If \( y(t) \) solves the IVP \( y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha \) and

\[
w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0w_{i+1-m}
\]

\[
h[b_{m-1}f(t_{i+1}, w_{i+1}) + b_{m-2}f(t_i, w_i) + \cdots + b_0f(t_{i+1-m}, w_{i+1-m})],
\]

the local truncation error is

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + \cdots + a_0y(t_{i+1-m})}{h} - [b_{m-1}f(t_{i+1}, y(t_{i+1})) + \cdots + b_0f(t_{i+1-m}, y(t_i))]
\]

NOTE: the local truncation error of a \( m \)-step explicit step is \( O(h^m) \).

the local truncation error of a \( m \)-step implicit step is \( O(h^{m+1}) \).
**m-step explicit step method vs. (m-1)-step implicit step method**

a) both have the same order of local truncation error, $O(h^m)$.

b) Implicit method usually has greater stability and smaller round-off errors.

For example, local truncation error of Adams-Bashforth 3-step explicit method, $\tau_{i+1}(h) = \frac{3}{8} y^{(4)}(\mu_i) h^3$.

Local truncation error of Adams-Moulton 2-step implicit method, $\tau_{i+1}(h) = -\frac{1}{24} y^{(4)}(\xi_i) h^3$.

**Predictor-Corrector Method**

*Motivation:* (1) Solve the IVP $y' = e^y$, $0 \leq t \leq 0.25$, $y(0) = 1$ by the three-step Adams-Moulton method.

Solution: The three-step Adams-Moulton method is

$$w_{i+1} = w_i + \frac{h}{24} [9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}]$$

Eq. (1)

Eq. (1) can be solved by Newton’s method. However, this can be quite computationally expensive.

(2) combine explicit and implicit methods.

**4th order Predictor-Corrector Method**

(we will combine 4th order Runge-Kutta method + 4th order 4-step explicit Adams-Bashforth method + 4th order three-step Adams-Moulton implicit method)

Step 1: Use 4th order Runge-Kutta method to compute $w_0, w_1, w_2$ and $w_3$.

Step 2: For $i = 3, 5, ... N$

a) Predictor sub-step. Use 4th order 4-step explicit Adams-Bashforth method to compute a predicated value

$$w_{i+1,p} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

b) Correction sub-step. Use 4th order three-step Adams-Moulton implicit method to compute a correction (the approximation at $i + 1$ time step)

$$w_{i+1} = w_i + \frac{h}{24} [9f(t_{i+1}, w_{i+1,p}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})]$$
5.10 Stability

Consistency and Convergence

**Definition.** A one-step difference equation with local truncation error $\tau_i(h)$ is said to be **consistent** if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| = 0$$

**Definition.** A one-step difference equation is said to be **convergent** if

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| = 0$$

where $y(t_i)$ is the exact solution and $w_i$ is the approximate solution.

**Example.** To solve $y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$. Let $|y''(t)| \leq M$, an $f(t, y)$ be continuous and satisfy a Lipschitz condition with Lipschitz constant $L$. Show that Euler’s method is consistent and convergent.

Solution:

$$|\tau_{i+1}(h)| = \left| \frac{h}{2} y''(\xi_i) \right| \leq \frac{h}{2} M$$

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| \leq \lim_{h \to 0} \frac{h}{2} M = 0$$

Thus Euler’s method is consistent.

By Theorem 5.9,

$$\max_{1 \leq i \leq N} |w_i - y(t_i)| \leq \frac{M h}{2L} [e^{L(b-a)} - 1]$$

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| \leq \lim_{h \to 0} \frac{M h}{2L} [e^{L(b-a)} - 1] = 0$$

Thus Euler’s method is convergent.

The rate of convergence of Euler’s method is $O(h)$. 
Motivation: How does round-off error affect approximation? To solve IVP \( y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \) by Euler’s method. Suppose \( \delta_i \) is the round-off error associated with each step.

\[
\begin{align*}
    u_0 &= \alpha + \delta_0 \\
    u_{i+1} &= u_i + hf(t_i, u_i) + \delta_{i+1} \quad \text{for each } i = 0, 1, ..., N-1.
\end{align*}
\]

Then \( |u_i - y(t_i)| \leq \frac{1}{L} \left( \frac{hM}{2} + \frac{\delta}{h} \right) \left[ e^{L(t_i-a)} - 1 \right] + |\delta_0| e^{L(t_i-a)}. \) Here \( |\delta_i| < \delta. \)

\[
\lim_{h \to 0} \left( \frac{hM}{2} + \frac{\delta}{h} \right) = \infty.
\]

**Stability:** small changes in the initial conditions produce correspondingly small changes in the subsequent approximations.

**Convergence of One-Step Methods**

Theorem. Suppose the IVP \( y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \) is approximated by a one-step difference method in the form

\[
\begin{align*}
    w_0 &= \alpha, \\
    w_{i+1} &= w_i + h\phi(t_i, w_i, h) \quad \text{where } i = 0, 2, \ldots, N.
\end{align*}
\]

Suppose also that \( h_0 > 0 \) exists and \( \phi(t, w, h) \) is continuous with a Lipschitz condition in \( w \) with constant \( L \) on \( D \), then

\[
D = \{(t, w, h) \mid a \leq t \leq b, -\infty < w < \infty, 0 \leq h \leq h_0\}.
\]

1. The method is **stable**;
2. The method is **convergent** if and only if it is **consistent**:

\[
\phi(t, w, 0) = f(t, y), \quad \text{for all } a \leq t \leq b
\]

3. If \( \tau \) exists s.t. \( |\tau_i(h)| \leq \tau(h) \) when \( 0 \leq h \leq h_0 \), then

\[
|w_i - y(t_i)| \leq \frac{\tau(h)}{L} e^{L(t_i-a)}.
\]