6.1 Linear Systems of Equations

To solve a system of linear equations

$$
\begin{aligned}
E_1:&\quad a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\
E_2:&\quad a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\
&\quad \vdots \\
E_n:&\quad a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n
\end{aligned}
$$

for $x_1, x_2, \dots, x_n$ by **Gaussian elimination with backward substitution**.

Matrix form:

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Three elementary row operations.

1. Multiply one row by a nonzero number: $(\lambda E_i) \to (E_i)$
2. Interchange two rows: : $(E_j) \leftrightarrow (E_i)$
3. Add a multiple of one row to a different row: $\left(E_i + \lambda E_j\right) \to (E_i)$

Echelon form (upper triangular form)

A matrix is in row-echelon form if

1. All rows consisting entirely of zeros are at the bottom
2. Each leading entry (first nonzero entry from left) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

$$
\begin{bmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{2} & -8 & 8 \\ 0 & 0 & 0 & \boxed{-9} \end{bmatrix}
\qquad\qquad
\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & -8 & 8 \\ 0 & 1 & 0 & -9 \end{bmatrix}
$$

Backward substitution

Example 1. To solve $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -5/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$

Solution: From $-\frac{5}{2}x_3 = -3$

$x_3 = \frac{6}{5}$.

Then from $2x_2 + x_3 = 4$

$2x_2 + \frac{6}{5} = 4$

$x_2 = \frac{7}{5}$.

Lastly from $x_1 + x_2 + 2x_3 = 6$

$x_1 + \frac{7}{5} + 2\left(\frac{6}{5}\right) = 6$

$x_1 = \frac{11}{5}$

Gaussian Elimination with Backward Substitution

1. Write the system of linear equations as an **augmented matrix** $[A \mid b]$.
2. Perform elementary row operations to put the augmented matrix in the echelon form
3. Solve the echelon form using backward substition

Example 2. Solve the system of linear equations

$$\begin{aligned} 2x_2 + x_3 &= 4 \\ x_1 + x_2 + 2x_3 &= 6 \\ 2x_1 + x_2 + x_3 &= 7 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right] \underset{(E_1) \leftrightarrow (E_2)}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right] \underset{(E_3 - 2 * E_1) \rightarrow (E_3)}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & -5 \end{array}\right] \underset{(E_3 + 0.5 * E_2) \rightarrow (E_3)}{\rightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -\frac{5}{2} & -3 \end{array}\right]$$

Now use backward substation to solve for values of x_1, x_2, x_3 (see **Example 1**).

Remark: Gaussian elimination is computationally expensive. The total number of multiplication and divisions is about $n^3/3$, where n is the number of unknowns.

6.2 Pivoting Strategies

Motivation: To solve a system of linear equations

$$\begin{aligned} E_1&: \; a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ E_2&: \; a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ &\vdots \\ E_n&: \; a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{aligned}$$

for $x_1, x_2, \dots, x_n$ by **Gaussian elimination** where $a_{kk}^{(k)}$ are numbers with small magnitude.

- In Gaussian elimination, if a pivot element $a_{kk}^{(k)}$ is small compared to an element $a_{jk}^{(k)}$ below, the multiplier $m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$ will be large, leading to large round-off errors.

Example 1. Apply Gaussian elimination to solve

$E1$: $\quad 0.003000x_1 + 59.14x_2 = 59.17$

$E2$: $\quad 5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding (The exact solution is $x_1 = 10.00, \;\; x_2 = 1.000$).

Ideas of Partial Pivoting.

Partial pivoting finds the smallest $p \geq k$ such that

$$\left|a_{pk}^{(k)}\right| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and interchanges the rows $(E_k) \leftrightarrow (E_p)$

Example 2. Apply Gaussian elimination with partial pivoting to solve

$E1$: $0.003000x_1 + 59.14x_2 = 59.17$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

Solution:

Step 1 of partial pivoting

$$\max\left\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\right\} = \{|0.003000|, |5.291|\} = 5.291 = |a_{21}^{(1)}|.$$

So perform $(E_1) \leftrightarrow (E_2)$ to make 5.291 the pivot element.

$$E1: \quad 5.291x_1 - 6.130x_2 = 46.78$$
$$E2: \quad 0.003000x_1 + 59.14x_2 = 59.17$$

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{0.003000}{5.29} = 0.0005670$$

Perform $\left(E_2 - m_{21}E_1\right) \rightarrow (E_2)$

$$5.291x_1 - 6.130x_2 = 46.78$$
$$59.14x_2 \approx 59.14$$

Backward substitution with 4-digit rounding leads to $x_1 = 10.00$; $x_2 = 1.000$.

Gaussian Elimination with Partial Pivoting (Algorithm 6.2)

$$\begin{array}{ll} E_1 & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\ E_2 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\ & \vdots \\ E_n & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1} \end{array}$$

INPUT: number of equations n; augmented matrix $A = [a_{ij}]$. Here $1 \le i \le N$; $1 \le j \le N + 1$

OUTPUT: solution $x_1,\ x_2,\ \ldots,\ x_n$

STEP 1 For $i = 1, \ldots, n$ set *NROW(i)=i*.

STEP 2 For $i = 1, \ldots, n-1$ do **STEP**s 3 – 6

STEP 3 Let p be the smallest integer with $i \le p \le n$ and

$$\left|a_{NROW(p),i}\right| = \max_{i \le j \le n} \left|a_{NROW(j),i}\right|.$$

STEP 4 If $a_{NROW(p),i} = 0$ then OUTPUT('no unique solution exists');

STOP.

STEP 5 If $NROW(i) \ne NROW(p)$ then set $NCOPY = NROW(i)$;

$NROW(i) = NROW(p)$;

$NROW(p) = NCOPY$.

STEP 6 For $j = i+1, \ldots, n$ do **STEP**s 7 and 8.

STEP 7 Set $m_{NROW(j),i} = \frac{a_{NROW(j),i}}{a_{NROW(j),i}}$

STEP 8 Perform $\left(E_{NROW(j)} - m_{NROW(j),i} E_{NROW(i)}\right) \rightarrow \left(E_{NROW(j)}\right)$

STEP 9 If $a_{NROW(n),n} = 0$ then OUTPUT('no unique solution exists');

STOP.

STEP 10 Set $x_n = a_{NROW(n),n+1} / a_{NROW(n)n}$ // *Start backward substitution*

STEP 11 For $i = n-1, \ldots, 1$

set $x_i = (a_{NROW(i),n+1} - \sum_{j=i+1}^{n} a_{NROW(i),j} x_j) / a_{NROW(i),i}$

STEP 12 OUTPUT($x_1, x_2, \ldots, x_n$);

STOP.

Example 3. Apply Gaussian elimination with partial pivoting to solve

$E1$: $30.00x_1 + 591400x_2 = 591700$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

Solution:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{5.291}{30.00} = 0.1764$$

$$30.00x_1 + 591400x_2 = 591700$$

$$-104300x_2 \approx -104400$$

Using backward substitution with 4-digit arithmetic leads to $x_1 = -10.00,\ x_2 = 1.001$.

Scaled Partial Pivoting

- If there are large variations in magnitude of the elements within a row, scaled partial pivoting should be used.
- Define a scale factor s_i for each row

$$s_i = \max_{1 \le j \le n} |a_{ij}|$$

- At step i, find p (the element which will be used as pivot) such that

$$\frac{a_{pi}}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k}$$ and interchange the rows $(E_i) \leftrightarrow (E_p)$

NOTE: The effect of scaling is to ensure that the largest element in each row has a relative magnitude of 1 before the comparison for row interchange is performed.

Example 3. Apply Gaussian elimination with scaled partial pivoting to solve

$E1$: $30.00x_1 + 591400x_2 = 591700$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

Solution:

$$s_1 = 591400$$

$$s_2 = 6.130$$

Consequently

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.5073 \times 10^{-4}$$

$$\frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631$$

5.291 should be used as pivot element. So $(E_1) \leftrightarrow (E_2)$

Solve
$$5.291x_1 - 6.130x_2 = 46.78$$
$$30.00x_1 + 591400x_2 = 591700$$

$x_1 = 10.00,\ x_2 = 1.000.$

Gaussian Elimination with Scaled Partial Pivoting (Algorithm 6.3)

The only steps in Alg. 6.3 that differ from those of Alg. 6.2 are:

STEP 1 For $i = 1, \dots, n$ set $s_i = \max_{1 \le j \le n} |a_{ij}|$;

If $s_i = 0$ then OUTPUT('no unique solution exists');

STOP.

set *NROW(i)=i.*

STEP 2 For $i = 1, \dots, n-1$ do **STEP**s 3 – 6

STEP 3 Let p be the smallest integer with $i \le p \le n$ and

$$\frac{\left|a_{NROW(p),i}\right|}{s_{NROW(p)}} = \max_{i \le j \le n} \frac{|a_{NROW(j),i}|}{s_{NROW(j)}}.$$