6.3 Linear Algebra and Matrix Inversion

Linear Algebra

Two matrices A and B are **equal** if they have same number of rows and columns $n \times m$ and if $a_{ij} = b_{ij}$.

If A and B are $n \times m$ matrices, **sum** $A + B$ is $n \times m$ matrix with entries $a_{ij} + b_{ij}$.

If A is $n \times m$ matrix and λ a real number, the **scalar multiplication** λA is $n \times m$ matrix with entries λa_{ij}.

Properties

Let A, B, C be $n \times m$ matrices, λ, μ real numbers.

(a) (commutative law) $A + B = B + A$
(b) (associative law) $(A + B) + C = A + (B + C)$
(c) $A + 0 = 0 + A = A$. Here 0 is $n \times m$ matrix with zero entries
(d) $A + (-A) = -A + A = 0$
(e) (distributive law of scale multiplication) $\lambda(A + B) = \lambda A + \lambda B$
(f) $\lambda(\mu A) = (\lambda\mu)A$
(g) $(\lambda + \mu)A = \lambda A + \mu A$
(h) $1A = A$

Matrix multiplication

Let A be $n \times m$ and B be $m \times p$. The matrix product $C = AB$ is $n \times p$ matrix with entries

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}$$

(Matrix vector multiplication can be viewed as a special case of matrix multiplication)

Special Matrices

- A *square* matrix has $m = n$
- A *diagonal* matrix $D = [d_{ij}]$ is square with $d_{ij} = 0$ when $i \neq j$.
- The *identity* matrix of order n, $I_n = [\delta_{ij}]$, is diagonal with $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$

- An *upper-triangular* $n \times n$ matrix $\mathrm{U} = \left[u_{ij}\right]$ has $u_{ij} = 0$, if $i = j + 1, \dots, n$.
- A *lower-triangular* $n \times n$ matrix $\mathrm{L} = \left[l_{ij}\right]$ has $l_{ij} = 0$, if $i = 1,2, \dots, j - 1$.

Theorem 6.8 Let A be $n \times m$, B be $m \times k$, C be $k \times p$, D be $m \times k$, and λ be a real number.

(a) $A(BC) = (AB)C$
(b) $A(B + D) = AB + AD$
(c) $I_m B = B$ and $BI_k = B$
(d) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

Matrix Inversion

- An $n \times n$ matrix A is *nonsingular* or *invertible* if $n \times n$ A^{-1} exists with $AA^{-1} = A^{-1}A = I$
- The matrix A^{-1} is called the *inverse* of A
- A matrix without an inverse is called singular or noninvertible

Theorem 6.12 For any nonsingular $n \times n$ matrix A,

(a) A^{-1} is unique
(b) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$
(c) If B is nonsingular $n \times n$, then $(AB)^{-1} = B^{-1}A^{-1}$

Matrix Transpose

- The *transpose* of $n \times m$ $A = [a_{ij}]$ is $m \times n$ $A^t = [a_{ji}]$
- A square matrix A is called *symmetric* if $A = A^t$

Theorem 6.14

(a) $(A^t)^t = A$
(b) $(A + B)^t = A^t + B^t$
(c) $(AB)^t = B^t A^t$
(d) If A^{-1} exists, then $(A^{-1})^t = (A^t)^{-1}$

6.4 Determinant of Matrix

(a) If $A = [a]$ is 1×1 matrix, then $\det A = a$

(b) If A is $n \times n$, the *minor* M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix by deleting row i and column j of A

(c) The *cofactor* $A_{ij} = (-1)^{i+j} M_{ij}$

(d) The determinant of $n \times n$ matrix A for $n > 1$ is

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Theorem 6.16 Let A be $n \times n$ matrix.

(a) If any row or column of A has all zeros, then $\det A = 0$
(b) If A has two rows or two columns equal, then $\det A = 0$
(c) If $\tilde{A}$ is obtained from A by $(E_i) \leftrightarrow (E_j)$, then $\det\tilde{A} = -\det A$
(d) If $\tilde{A}$ is obtained from A by $(\lambda E_i) \rightarrow (E_i)$, then $\det\tilde{A} = \lambda \det A$
(e) If $\tilde{A}$ is obtained from A by $\left(E_i + \lambda E_j\right) \rightarrow (E_i)$, then $\det\tilde{A} = \det A$
(f) If B is $n \times n$, then $\det(AB) = \det A \det \mathrm{B}$
(g) $\det A^t = \det A$
(h) When A^{-1} exists, $\det A^{-1} = 1/(\det A)$
(i) If A is *upper/lower triangular* or *diagonal* matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$

Theorem 6.17 The following statements are equivalent for any $n \times n$ matrix A:

(a) The equation $A\boldsymbol{x} = \mathbf{0}$ has unique solution $\boldsymbol{x} = \mathbf{0}$
(b) The system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $\boldsymbol{b}$
(c) The matrix A is nonsingular
(d) $\det A \neq 0$
(e) Gaussian elimination with row interchanges can be performed on $A\boldsymbol{x} = \boldsymbol{b}$ for any $\boldsymbol{b}$

6.5 Matrix Factorization

Motivation: Consider to solve $A\boldsymbol{x} = \boldsymbol{b}$. Here A is $n \times n$ matrix. Suppose $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

First solve $L\boldsymbol{y} = \boldsymbol{b}$

Then solve $U\boldsymbol{x} = \boldsymbol{y}$

Consider the first step of Gaussian elimination (assume no row interchange) on $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Do $(E_j - m_{j1}E_1) \to (E_j)$ for $j = 2,3, \dots, n$. Here $m_{j1} = \frac{a_{ji}}{a_{11}}$ to obtain

$$A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix}$$

Note: $a_{11}^{(1)} = a_{11},\ a_{12}^{(1)} = a_{12}, \dots a_{1n}^{(1)} = a_{1n}$.

This is equivalent to

$$A^{(1)} = M^{(1)}A$$

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & \cdots & 1 \end{bmatrix}$$

$M^{(1)}$ is called the **first Gaussian transformation matrix**.

Similarly, the **kth Gaussian transformation matrix is**

$$M^{(k)} = \begin{bmatrix} 1 & 0 & & \cdots & \cdots & 0 \\ 0 & \ddots & & & & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & 0 & \ddots & \vdots & \vdots \\ \vdots & & \vdots & -m_{k+1,k} & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -m_{n,k} & 0 & \cdots 1 \end{bmatrix}$$

Gaussian elimination (without row interchange) can be written as
$A^{(n)} = M^{(n-1)}M^{(n-2)} \dots M^{(1)}A$ with

$$A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{(n)} \end{bmatrix}$$

LU Factorization $A = LU$

Reversing the elimination steps gives the inverses:

$$L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix} 1 & 0 & & \cdots & \cdots & 0 \\ 0 & \ddots & & & & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & 0 & \ddots & \vdots & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & m_{n,k} & 0 & \cdots 1 \end{bmatrix}$$

We define $A = LU = [M^{(n-1)}M^{(n-2)} \dots M^{(1)}]^{-1}A^{(n)}$

Here $U = A^{(n)}$ is the **upper triangular** matrix.

$L = [M^{(n-1)}M^{(n-2)} \dots M^{(1)}]^{-1} = [M^{(1)}]^{-1}[M^{(2)}]^{-1} \dots [M^{(n-1)}]^{-1}$ is the **lower triangular** matrix.

Theorem 6.19 If Gaussian elimination can be performed on the linear system $A\boldsymbol{x} = \boldsymbol{b}$ *without row interchange*, A can be factored into the product of *lower triangular* matrix L and *upper triangular* matrix U as $A = LU$:

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{(n)} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & \dots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

Example. Consider matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, which is obtained by interchanging the 2nd and 3rd rows of identity matrix $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. What is PA?

Permutation Matrices

Definition. Suppose $k_1, k_2, \dots, k_n$ is a permutation of $1,2,\dots,n$. The permutation matrix $P = [p_{ij}]$ is defined by

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i \\ 0 & \text{otherwise} \end{cases}$$

- PA permutes the rows of A:

$$PA = \begin{bmatrix} a_{k_1 1} & \dots & a_{k_1 n} \\ \vdots & \ddots & \vdots \\ a_{k_n 1} & \cdots & a_{k_n n} \end{bmatrix}$$

- P^{-1} exists and $P^{-1} = P^t$

Gaussian elimination with row interchanges can be written as:

$$A = P^{-1}LU = (P^tL)U$$

Remark: P^tL is not lower triangular matrix unless P is identity matrix.

Example. Find a factorization $A = (P^tL)U$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

Solution $(E_1) \leftrightarrow (E_2)$, then $(E_3 + E_1) \rightarrow (E_3)$ and $(E_4 - E_1) \rightarrow (E_4)$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$(E_2) \leftrightarrow (E_4)$ then $(E_4 + E_3) \rightarrow (E_4)$

$$U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Two row interchanges ($(E_1) \leftrightarrow (E_2)$ and $(E_2) \leftrightarrow (E_4)$)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad and \quad PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Gaussian elimination is performed on PA without row interchanges.

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU$$

Since $P^{-1} = P^t$,

$$A = P^{-1}LU = (P^tL)U = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$