6.6 Special Types of Matrices

Definition. The $n \times n$ matrix A is said to be *strictly diagonally dominant* when $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ holds for each $i = 1,2,3, \ldots, n$

Example. Determine if matrices $A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}$, and $B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & -1 \\ 0 & 5 & -6 \end{bmatrix}$ are strictly diagonally dominant.

Theorem 6.21. A strictly diagonally dominant matrix A is nonsingular, Gaussian elimination can be performed on $A\boldsymbol{x} = \boldsymbol{b}$ without row interchanges. The computations will be stable with respect to the growth of round-off errors.

Definition. Matrix A is said to be *positive definite* if it is symmetric and if $\boldsymbol{x}^t A \boldsymbol{x} > 0$ for every nonzero vector $\boldsymbol{x}$ (i.e. $\boldsymbol{x} \neq \boldsymbol{0}$)

Example. Show that the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite.

Solution: Let $\boldsymbol{x} = [x_1, x_2, x_3]^{\boldsymbol{t}}$ be a 3-dimentional column vector.

$$\boldsymbol{x}^t A \boldsymbol{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$$

$$= x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2 = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

unless $x_1 = x_2 = x_3 = 0$.

Theorem 6.23. If A is an $n \times n$ positive definite matrix, then

(i) A has an inverse
(ii) $a_{ii} > 0$ for each $i = 1,2,3, \ldots, n$
(iii) $\max_{1 \leq k,j \leq n} |a_{kj}| \leq \max_{1 \leq i \leq n} |a_{ii}|$
(iv) $(a_{ij})^2 < a_{ii}a_{jj}$ for each $i \neq j$

Definition. A *leading principal submatrix* of a matrix A is a matrix of the form $A_k = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix}$ for some $1 \le k \le n$.

Theorem 6.25. A symmetric matrix A is *positive definite* if and only if each of its *leading principal submatrix* has a positive determinant.

Example. Show all leading principle submatrix of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ have positive determinants.

Symmetric positive definite matrix and Gaussian elimination

Theorem 6.26. The symmetric matrix A is positive definite if and only if Gaussian elimination without row interchanges can be done on $A\boldsymbol{x} = \boldsymbol{b}$ with all pivot elements positive, and the computations are stable.

Corollary 6.27. The matrix A is positive definite if and only if A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries.

Corollary 6.28. The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries.

Definition. An $n \times n$ matrix is called a band matrix if integers p, q exist with $1 < p, q < n$ and $a_{ij} = 0$ when $p \leq j - i$ or $q \leq i - j$. The bandwidth is $w = p + q - 1$.

Tridiagonal matrices. $p = q = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & \dots & 0 \\ a_{21} & a_{22} & a_{23} & & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \dots & \dots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}$$

Theorem 6.31. Suppose $A = [a_{ij}]$ is tridiagonal with $a_{i,i-1}a_{i,i+1} \neq 0$. If $|a_{11}| > |a_{12}|$, $|a_{ii}| \geq |a_{i,i-1}| + |a_{i,i+1}|$, and $|a_{nn}| > |a_{n,n-1}|$, then A is nonsingular.