The Jacobi Method

Two assumptions made on Jacobi Method:

1. The system given by
\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]
has a unique solution.

2. The coefficient matrix $A$ has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \ldots, a_{nn}$ are nonzeros.

Main idea of Jacobi
To begin, solve the $1^{\text{st}}$ equation for $x_1$, the $2^{\text{nd}}$ equation for $x_2$ and so on to obtain the rewritten equations:

\[
\begin{align*}
  x_1 &= \frac{1}{a_{11}} (b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\
  x_2 &= \frac{1}{a_{22}} (b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\
  &\vdots \\
  x_n &= \frac{1}{a_{nn}} (b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})
\end{align*}
\]

Then make an initial guess of the solution $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the first approximation, $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)})$.

This accomplishes one iteration.

In the same way, the second approximation $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_n^{(2)})$ is computed by substituting the first approximation’s $x$-values into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)})$, $k = 1, 2, 3, \ldots$
The Jacobi Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $x^{(k)}$ from $x^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[ \sum_{j=1, \ j \neq i}^{n} (-a_{ij} x_j^{(k-1)}) + b_i \right], \quad \text{for } i = 1, 2, \ldots, n$$

Example. Apply the Jacobi method to solve

\begin{align*}
5x_1 - 2x_2 + 3x_n &= -1 \\
-3x_1 + 9x_2 + x_n &= 2 \\
2x_1 - x_2 - 7x_n &= 3
\end{align*}

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution. To begin, rewrite the system

\begin{align*}
x_1 &= -\frac{1}{5} + \frac{2}{5} x_2 - \frac{3}{5} x_3 \\
x_2 &= \frac{2}{9} + \frac{3}{9} x_1 - \frac{1}{9} x_3 \\
x_3 &= -\frac{3}{7} + \frac{2}{7} x_1 - \frac{1}{7} x_2
\end{align*}

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

\begin{align*}
x_1^{(1)} &= -\frac{1}{5} + \frac{2}{5} (0) - \frac{3}{5} (0) = -0.200 \\
x_2^{(1)} &= \frac{2}{9} + \frac{3}{9} (0) - \frac{1}{9} (0) = 0.222 \\
x_3^{(1)} &= -\frac{3}{7} + \frac{2}{7} (0) - \frac{1}{7} (0) = -0.429
\end{align*}
Continue iteration, we obtain

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>k = 0</th>
<th>k = 1</th>
<th>k = 2</th>
<th>k = 3</th>
<th>k = 4</th>
<th>k = 5</th>
<th>k = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^{(k)}$</td>
<td>0.000</td>
<td>-0.200</td>
<td>0.146</td>
<td>0.192</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_2^{(k)}$</td>
<td>0.000</td>
<td>0.222</td>
<td>0.203</td>
<td>0.328</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_2^{(k)}$</td>
<td>0.000</td>
<td>-0.429</td>
<td>-0.517</td>
<td>-0.416</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**The Jacobi Method in Matrix Form**

Consider to solve an $n \times n$ size system of linear equations $Ax = b$ with $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ for $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

We split $A$ into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n,n-1} & \cdots & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{n-1,n} \end{bmatrix} = D - L - U$$

$Ax = b$ is transformed into $(D - L - U)x = b$

$$Dx = (L + U)x + b$$

Assume $D^{-1}$ exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$x = D^{-1}(L + U)x + D^{-1}b$$
The matrix form of Jacobi iterative method is

\[
x^{(k)} = D^{-1}(L + U)x^{(k-1)} + D^{-1}b \quad k = 1, 2, 3, \ldots
\]

Define \( T = D^{-1}(L + U) \) and \( c = D^{-1}b \), Jacobi iteration method can also be written as

\[
x^{(k)} = Tx^{(k-1)} + c \quad k = 1, 2, 3, \ldots
\]

Numerical Algorithm of Jacobi Method

Input: \( A = [a_{ij}], \ b, xo = x^{(0)} \), tolerance \( TOL \), maximum number of iterations \( N \).

Step 1  Set \( k = 1 \)

Step 2  while \( (k \leq N) \) do Steps 3-6

   Step 3  For for \( i = 1, 2, \ldots n \)
   \[
x_i = \frac{1}{a_{ii}} \left[ \sum_{j=1, j \neq i}^{n} (-a_{ij}x_j) + b_i \right],
   \]

   Step 4  If \( ||x - xo|| < TOL \), then OUTPUT \( (x_1, x_2, x_3, \ldots x_n) \);  
          STOP.

   Step 5  Set \( k = k + 1 \).

   Step 6  For for \( i = 1, 2, \ldots n \)
   Set \( xo_i = x_i \).

   Step 7  OUTPUT \( (x_1, x_2, x_3, \ldots x_n) \);  
          STOP.

Another stopping criterion in Step 4: \( \frac{||x^{(k)} - x^{(k-1)}||}{||x^{(k)}||} \)
The Gauss-Seidel Method

Main idea of Gauss-Seidel

With the Jacobi method, the values of $x_i^{(k)}$ obtained in the $k$th iteration remain unchanged until the entire $(k + 1)$th iteration has been calculated. With the Gauss-Seidel method, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example, once we have computed $x_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(k+1)}$, and so on.

Example. Derive iteration equations for the Jacobi method and Gauss-Seidel method to solve

\[
\begin{align*}
5x_1 - 2x_2 + 3x_n &= -1 \\
-3x_1 + 9x_2 + x_n &= 2 \\
2x_1 - x_2 - 7x_n &= 3
\end{align*}
\]

The Gauss-Seidel Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from $\mathbf{x}^{(k-1)}$ by

\[
x_i^{(k)} = \frac{1}{a_{ii}} \left[ - \sum_{j=1}^{i-1} (a_{ij} x_j^{(k)}) - \sum_{j=i+1}^{n} (a_{ij} x_j^{(k-1)}) + b_i \right], \quad \text{for } i = 1, 2, ..., n
\]

Namely,

\[
\begin{align*}
a_{11} x_1^{(k)} &= -a_{12} x_2^{(k-1)} - \cdots - a_{1n} x_n^{(k-1)} + b_1 \\
-a_{21} x_1^{(k)} + a_{22} x_2^{(k)} &= -a_{23} x_3^{(k-1)} - \cdots - a_{2n} x_n^{(k-1)} + b_2 \\
&\quad \vdots \\
-a_{n1} x_1^{(k)} + a_{n2} x_2^{(k)} + \cdots a_{nn} x_n^{(k)} &= b_n
\end{align*}
\]

Matrix form of Gauss-Seidel method.

\[
(D - L) \mathbf{x}^{(k)} = U \mathbf{x}^{(k-1)} + \mathbf{b}
\]

\[
\mathbf{x}^{(k)} = (D - L)^{-1} U \mathbf{x}^{(k-1)} + (D - L)^{-1} \mathbf{b}
\]

Define $T_g = (D - L)^{-1} U$ and $c_g = (D - L)^{-1} \mathbf{b}$, Gauss-Seidel method can be written as

\[
\mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g \quad k = 1, 2, 3, ...
\]
**Numerical Algorithm of Gauss-Seidel Method**

Input: \( A = [a_{ij}] \), \( b, X_0 = x^{(0)} \), tolerance \( TOL \), maximum number of iterations \( N \).

Step 1 Set \( k = 1 \)

Step 2 while \((k \leq N)\) do Steps 3-6

   Step 3 For for \( i = 1, 2, \ldots, n \)

       \[ x_i = \frac{1}{a_{ii}} \left[ - \sum_{j=1}^{i-1} (a_{ij}x_j) - \sum_{j=i+1}^{n} (a_{ij}X_O) + b_i \right]. \]

   Step 4 If \( ||x - X_O|| < TOL\), then OUTPUT \((x_1, x_2, x_3, \ldots, x_n)\);

   Stop.

   Step 5 Set \( k = k + 1 \).

   Step 6 For for \( i = 1, 2, \ldots, n \)

       Set \( X_O_i = x_i \).

   Step 7 OUTPUT \((x_1, x_2, x_3, \ldots, x_n)\);

   Stop.

**Convergence theorems of the iteration methods**

Let the iteration method be written as

\[ x^{(k)} = Tx^{(k-1)} + c \]

for each \( k = 1, 2, 3, \ldots \)

**Lemma 7.18** If the spectral radius satisfies \( \rho(T) < 1 \), then \( (I-T)^{-1} \) exists, and

\[ (I-T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j \]

**Theorem 7.19** For any \( x^{(0)} \in R^n \), the sequence \( \{x^{(k)}\}_{k=0}^{\infty} \) defined by
\[ x^{(k)} = Tx^{(k-1)} + c \quad \text{for each } k \geq 1 \]
converges to the unique solution of \( x =Tx +c \) if and only if \( \rho(T) < 1 \).

**Proof** (only show \( \rho(T) < 1 \) is sufficient condition)

\[ x^{(k)} = Tx^{(k-1)} + c = T(Tx^{(k-2)} + c) + c = \ldots = T^k x^{(0)} + (T^{k-1} + \ldots + T + I)c \]

Since \( \rho(T) < 1 \), \( \lim_{k \to \infty} T^k x^{(0)} = 0 \)

\[
\lim_{k \to \infty} x^{(k)} = 0 + \lim_{k \to \infty} \left( \sum_{j=0}^{k-1} T^j \right) c = (I - T)^{-1} c
\]

**Corollary 7.20** If \( ||T|| < 1 \) for any natural matrix norm and \( c \) is a given vector, then the sequence \( \{x^{(k)}\}_{k=0}^\infty \) defined by \( x^{(k)} = Tx^{(k-1)} + c \) converges, for any \( x^{(0)} \in R^n \), to a vector \( x \in R^n \), with \( x = Tx + c \), and the following error bound hold:

(i) \( ||x - x^{(k)}|| \leq ||T||^k ||x^{(0)} - x|| \)

(ii) \( ||x - x^{(k)}|| \leq \frac{||T||^k}{1-||T||} ||x^{(1)} - x^{(0)}|| \)

**Theorem 7.21** If \( A \) is strictly diagonally dominant, then for any choice of \( x^{(0)} \), both the Jacobi and Gauss-Seidel methods give sequences \( \{x^{(k)}\}_{k=0}^\infty \) that converges to the unique solution of \( Ax = b \).

**Rate of Convergence**

**Corollary 7.20** (i) implies \( ||x - x^{(k)}|| \approx \rho(T)^k ||x^{(0)} - x|| \)

**Theorem 7.22** (Stein-Rosenberg) If \( a_{ij} \leq 0 \), for each \( i \neq j \) and \( a_{ii} \geq 0 \), for each \( i = 1, 2, \ldots, n \), then one and only one of following statements holds:

(i) \( 0 \leq \rho(T_g) < \rho(T_j) < 1; \)

(ii) \( 1 < \rho(T_j) < \rho(T_g); \)

(iii) \( \rho(T_j) = \rho(T_g) = 0; \)

(iv) \( \rho(T_j) = \rho(T_g) = 1. \)