1.1 Review of Calculus

Limits and Continuity

Definition

A function f defined on a set X of real numbers has the *limit* L at x_0, written $\lim_{x \to x_0} f(x) = L$, if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \quad \text{whenever} \quad x \in X \text{ and } 0 < |x - x_0| < \delta.$$

Definition

Let f be a function defined on a set X of real numbers and $x_0 \in X$. Then f is *continuous* at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The function f is continuous on the set X if it is continuous at each number in X.

Limits of Sequence

Definition

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real of complex numbers. The sequence $\{x_n\}_{n=1}^{\infty}$ has the *limit* x is, for any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$, whenever $n > N(\varepsilon)$. The notation

$$\lim_{n\to\infty} x_n = x, \text{ or } x_n \to x \text{ as } n \to \infty,$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Theorem

If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

1. *f is continuous at x_0;*
2. *If the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to x_0, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

Derivative

Definition

Let f be a functions defined in an open interval containing x_0. The function f is *differentiable* at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the *derivative* of f at x_0. A function that has a derivative at each number in a set X is *differentiable* on X.

- **Theorem**

If the function f is differentiable at x_0, then f is continuous at x_0.

$C[a,b]$: set of all functions continuous on interval $[a,b]$.

- **Rolle's Theorem**

Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then a number c in (a,b) exists with $f'(c) = 0$.

- **Mean Value Theorem**

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

- **Extreme Value Theorem**

If $f \in C[a,b]$, then $c_1, c_2 \in [a,b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a,b]$. Additionally, if f is differentiable on (a,b), then the numbers $c_1\ and\ c_2$ occur either at the endpoints of $[a,b]$ or where f' is zero.

Taylor's Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$ and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

Where

$$P_n(x)$$
$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

- $P_n(x)$ – nth Taylor polynomial
- $R_n(x)$ – remainder term (truncation error)
- When $x_0 = 0$, $P_n(x)$ is also called Maclaurin polynomial

Example 3. Let $f(x) = \cos(x)$ and $x_0 = 0$. Determine

(a) the 2nd Taylor polynomial $P_2(x)$ for f about x_0, find a bound for the accuracy of the approximation;

(b) the 3rd Taylor polynomial $P_3(x)$ for f about x_0, find a bound for the accuracy of the approximation;

(c) use the third Taylor polynomial and its remainder term found in (b) to approximate $\int_0^{0.1} \cos(x)\,dx$.

Exercise 1.1.18. Let $f(x) = (1 - x)^{-1}$ and $x_0 = 0$. Find the nth Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a necessary n for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on [0, 0.5].