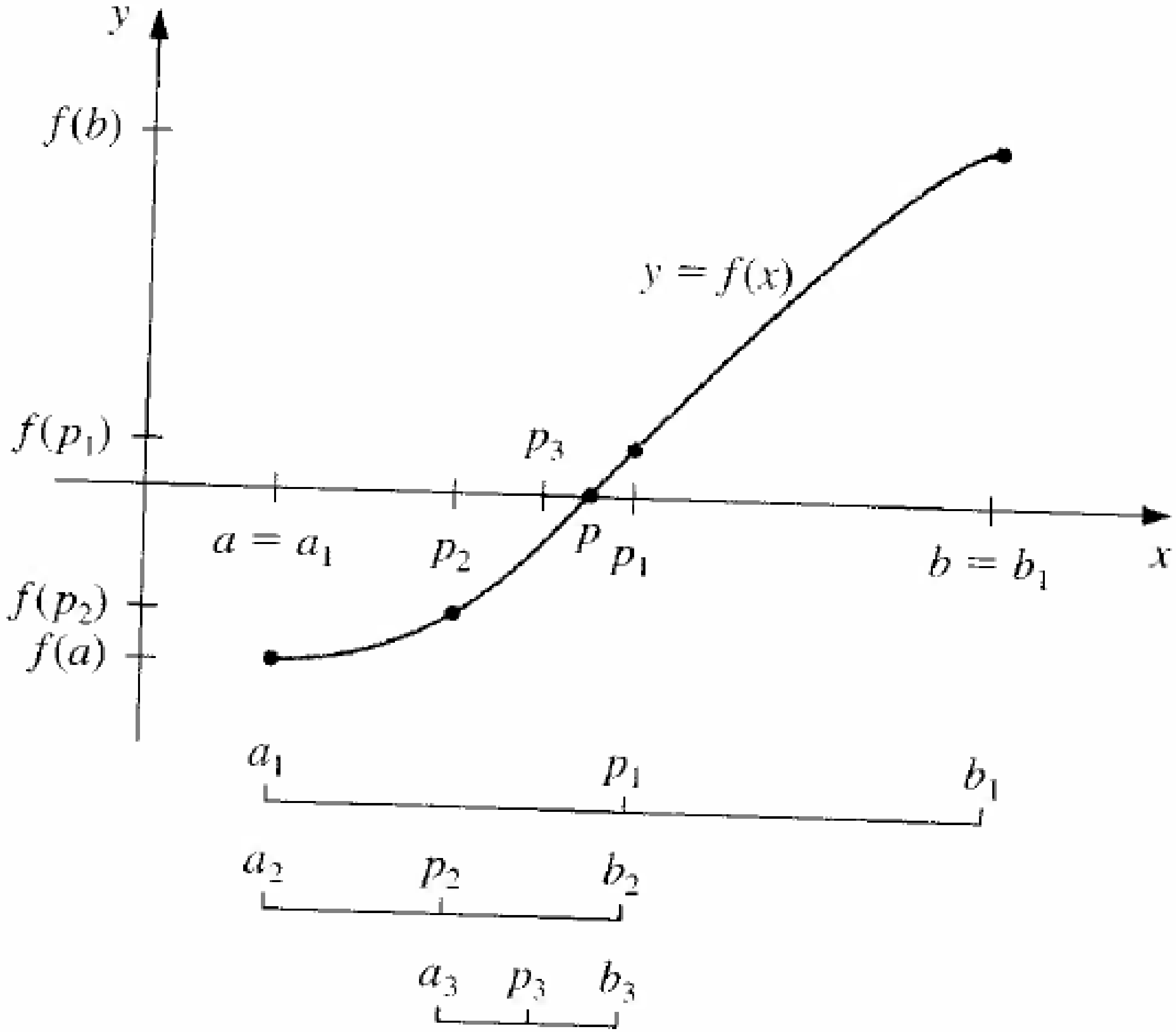
2.1 The Bisection Method


y
f(b)
y = f(x)
f(p1)
p3
a = a1
p2
p
p1
b = b1
x
f(p2)
f(a)
a1
p1
b1
a2
p2
b2
a3
p3
b3

Facts to remember:

1. The sequence of intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ contains the desired root.

2. Intervals containing the root: $(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset (a_4, b_4) \ldots$

3. After n steps, the interval (a_n, b_n) has the length: $b_n - a_n = (1/2)^{n-1}(b - a)$

4. Let $p_n = \frac{b_n + a_n}{2}$ be the mid-point of (a_n, b_n). The limit of sequence $\{p_n\}_{n=1}^{\infty}$ is the root.

Convergence

- **Theorem**

Suppose function $f(x)$ is continuous on $[a, b]$, and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of $f(x)$ with

$$|p_n - p| \leq \left({}^{1}/_{2}\right)^{n}(b - a), \qquad \text{when } n \geq 1$$

- **Convergence rate**

The sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with the rate of convergence $O(({}^{1}/_{2})^{n})$:

$$p_n = p + O\left(\left({}^{1}/_{2}\right)^{n}\right)$$

Example 2.1.1. Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in [1, 2], and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Remark: $|p_n - p| \leq (1/2)^n (b - a)$

or $|p_n - p| \leq (1/2) \quad (b_n - a_n)$

- **Example 2.1.2.** Determine the number of iteration to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3}. Use $a_1 = 1, b_1 = 2$.

Solution: Since $|p_n - p| \leq (^1/_2)^n (b_1 - a_1) \leq 10^{-3}, \rightarrow$ $2^{-n}(2-1) \leq 10^{-3}$.

Solve for $n \rightarrow n \approx 9.96$.

So $n = 10$ is needed.

- **Exercise 2.1.13.** Find an approximation to $\sqrt[3]{25}$

Correct within 10^{-4} using bisection method.

Solution: Consider to solve $f(x) = x^3 - 25 = 0$ by the Bisection method.

By trial and error, we can choose $a_1 = 2, b_1 = 3$.

Because $f(a_1) \cdot f(b_1) < 0$.

The Algorithm

```
INPUT       a,b; tolerance TOL; maximum number of iterations N0.
OUTPUT      solution p or message of failure.
STEP1       Set  i = 1;
            FA = f(a);
STEP2       While i ≤ N0 do STEPs 3-6.
            STEP3 Set p = a + (b-a)/2;     // a good way of computing middle point
                      FP = f(p).
            STEP4  IF FP = 0 or (b-a) < TOL then
                      OUTPUT (p);
                      STOP.
            STEP5     Set i = i +1.
            STEP6     If FP·FA > 0 then
                        Set a = p;
                        FA = FP.
                      else
                        set b = p;
STEP7 OUTPUT("Method failed after N0 iterations");
        STOP.
```

Matlab Code

```
function p=bisection(f,a,b,tol)

while 1
   p=(a+b)/2;
   if p-a<tol, break; end
   if f(a)*f(p)>0
      a=p;
   else
      b=p;
   end
end   %while 1
```