

2.3 Newton's Method and Its Extension

Basic Idea

- Taylor's Theorem Recap

Suppose $f \in C^2[a, b]$ and $p_0 \in [a, b]$ approximates solution p of $f(x) = 0$ with $f'(p_0) \neq 0$. Expand $f(x)$ about p_0 :

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p))$$

$f(p) = 0$, and assume $(p - p_0)^2$ is negligible:

$$0 \approx f(p_0) + (p - p_0)f'(p_0)$$

Solving for p yields:

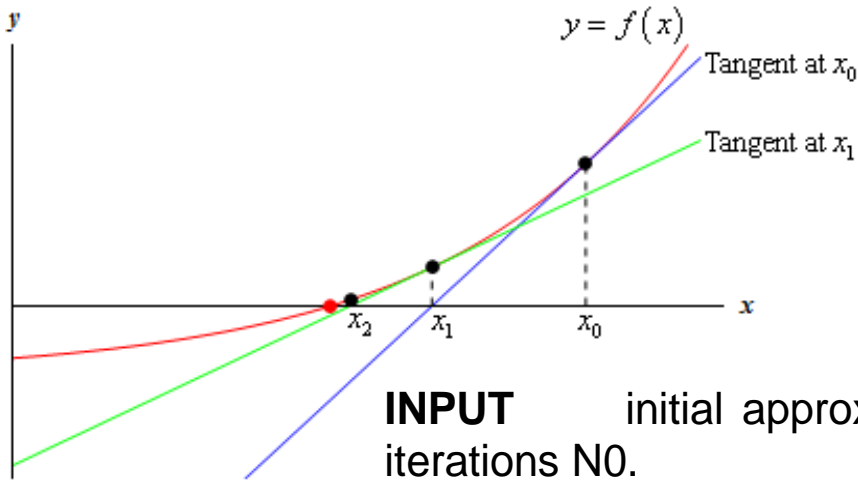
$$p \approx p_1 \equiv p_0 - \frac{f(p_0)}{f'(p_0)}$$

This gives the sequence $\{p_n\}_{n=0}^{\infty}$:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Remark: p_n is an improved approximation.

Algorithm: Newton's Method



INPUT initial approximation p_0 ; tolerance TOL; maximum number of iterations N_0 .

OUTPUT approximate solution p or message of failure.

STEP1 Set $i = 1$.

STEP2 While $i \leq N_0$ do STEPs 3-6

STEP3 Set $p = p_0 - f(p_0)/f'(p_0)$.

STEP4 If $|p - p_0| < \text{TOL}$ then

OUTPUT (p);

STOP.

STEP5 Set $i = i + 1$.

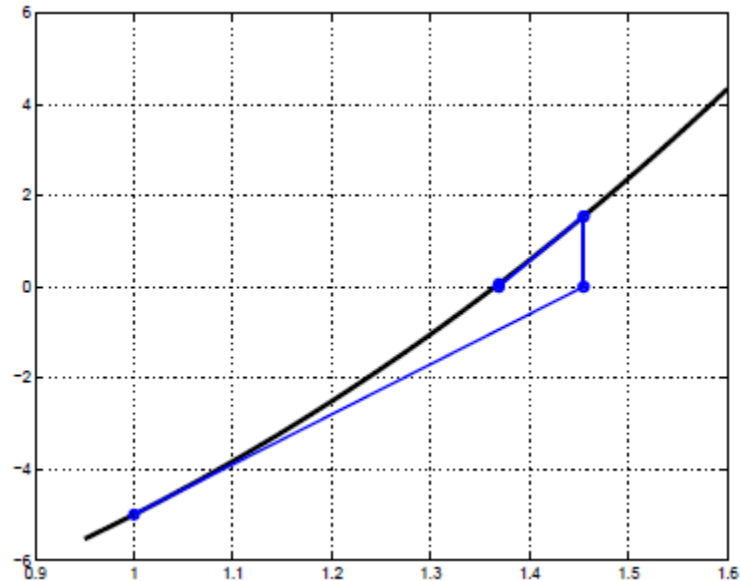
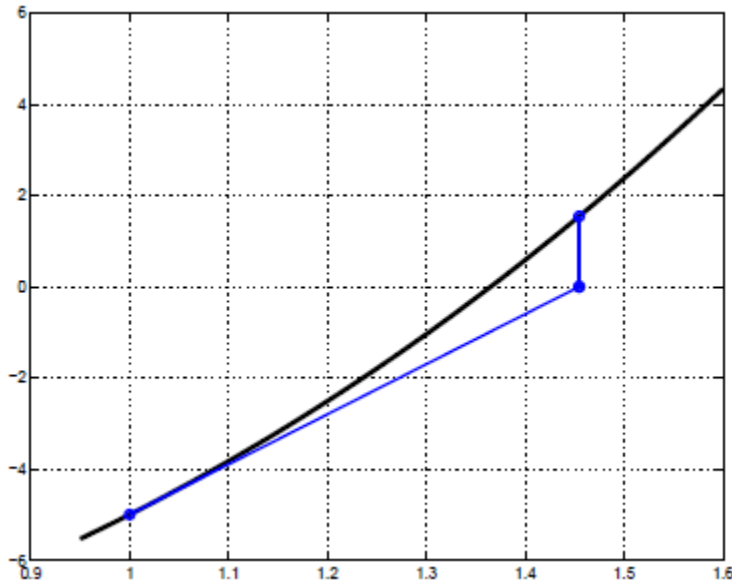
STEP6 Set $p_0 = p$.

STEP7 OUTPUT('The method failed');

STOP.

Geometric Interpretation

- Two steps of Newton's method for solving $f(x) = x^3 + 4x^2 - 10 = 0$.



- $p_0 = 1$
- $p_1 = p_0 - \frac{p_0^3 + 4p_0^2 - 10}{3p_0^2 + 8p_0} = 1.4545454545$
- $p_2 = p_1 - \frac{p_1^3 + 4p_1^2 - 10}{3p_1^2 + 8p_1} = 1.3689004011$

Example 2.3.1 Consider the function $f(x) = \cos(x) - x = 0$. Approximate a root of f using
(a) a fixed-point method, and
(b) Newton's method.

About Newton's Method

- Pros.

1. Fast convergence: Newton's method converges fastest among methods we explore (quadratic convergence)

- Cons.

1. $f'(x_{n-1})$ cause problems

Remark: Newton's method works best if $f' \geq k > 0$

2. Expensive: Computing derivative in every iteration

- We assume $|p - p_0|$ is small, then $|p - p_0|^2 \ll |p - p_0|$, and we can neglect the 2nd order term in Taylor expansion.

Remark: In order for Newton's method to converge we need a **good starting guess**.

Convergence

Relation to fixed-point iteration

Newton's method is fixed-point iteration with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Theorem 2.6

Let $f \in C^2[a, b]$ and $p \in [a, b]$ is that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

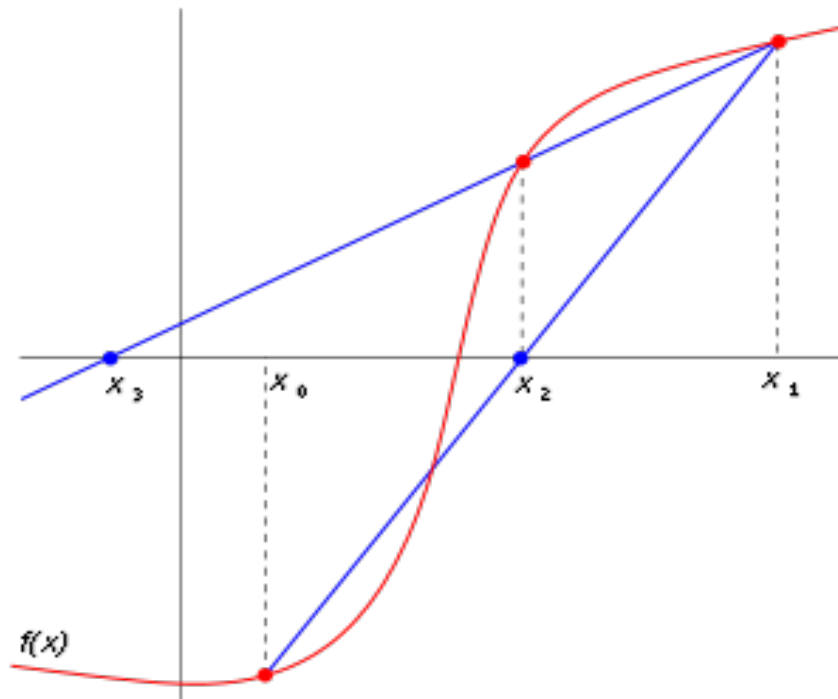
The Secant Method

- Approximate the derivative:

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$

to get

$$p_n = p_{n-1} - \frac{f(p_{n-1}) (p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \quad (2.12)$$



Summary of Secant method algorithm:

1. Make two initial guesses: p_0 and p_1
2. Use Eq. (2.12) to construct $p_2, p_3, p_4 \dots$ till accuracy is met.

Exercise 2.3.6 Consider the function $f(x) = e^x + 2^{-x} + 2\cos(x) - 6 = 0$. Approximate a root of f using the Secant method.

Algorithm: The Secant Method

INPUT initial approximation p_0, p_1 ; tolerance TOL; maximum number of iterations N_0 .

OUTPUT approximate solution p or message of failure.

STEP1 Set $i = 2$;

$q_0 = f(p_0)$;

$q_1 = f(p_1)$;

STEP2 While $i \leq N_0$ do STEPs 3-6

STEP3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$.

STEP4 If $|p - p_1| < \text{TOL}$ then

 OUTPUT (p);

 STOP.

STEP5 Set $i = i + 1$.

 STEP6 Set $p_0 = p_1$;

$q_0 = q_1$;

$p_1 = p$;

$q_1 = f(p)$.

STEP7 OUTPUT('The method failed');

STOP.

The Method of False Position

- The bisection method iterations satisfy:
 $|p_n - p| < \frac{1}{2} |a_n - b_n|$, which means the root lies between a_n and b_n .
- Root bracketing is not guaranteed for either Newton's method or Secant method.
- Method of false position: generate approximations in the same manner as the Secant method, but also includes a test to ensure that the root is always bracketed between successive iterations.

- Start with two points a_n, b_n which bracket the root, i.e, $f(a_n) \cdot f(b_n) < 0$. Let p_{n+1} be the zero-crossing of the secant line:

$$p_{n+1} = b_n - \frac{f(b_n)(a_n - b_n)}{f(a_n) - f(b_n)}$$

- Update as in the bisection method:
 If $f(a_n) \cdot f(p_{n+1}) > 0$, then $a_{n+1} = p_{n+1}, b_{n+1} = b_n$
 If $f(a_n) \cdot f(p_{n+1}) < 0$, then $a_{n+1} = a_n, b_{n+1} = p_{n+1}$

Remark: False position method is rarely used.