2.4 Error Analysis for Iterative Methods

Definition

- **Order of Convergence**

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n \; - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=0}^{\infty}$ is said to **converges to $\boldsymbol{p}$ of order $\boldsymbol{\alpha}$ with asymptotic error constant $\boldsymbol{\lambda}$.**

An iterative technique $p_n = g(p_{n-1})$ is said to be of order $\boldsymbol{\alpha}$ if the sequence $\{p_n\}_{n=0}^{\infty}$ converges to the solution $p = g(p)$ of order $\boldsymbol{\alpha}$.

- Special cases
  1. If $\alpha = 1$ (and $\lambda < 1$), the sequence is **linearly convergent**
  2. If $\alpha = 2$, the sequence is **quadratically convergent**
  3. If $\alpha < 1$, the sequence is **sub-linearly convergent** (undesirable, very slow)
  4. If $\alpha = 1$ and $\lambda = 0$ or $1 < \alpha < 2$, the sequence is **super-linearly convergent**

- **Remark:**

  High order (α) $\Longrightarrow$ faster convergence (more desirable)

  λ is less important than the order (α)

Linear vs. Quadratic

Suppose we have two sequences converging to 0 with:

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n\ |} = 0.9, \qquad \lim_{n\to\infty} \frac{|q_{n+1}|}{|q_n\ |^2} = 0.9$$

Roughly we have:

$$|p_n| \approx 0.9|p_{n-1}| \approx \cdots \approx 0.9^n|p_0|,$$
$$|q_n| \approx 0.9|q_{n-1}|^2 \approx \cdots \approx 0.9^{2^n-1}|q_0|,$$

Assume $p_0 = q_0 = 1$

n	p_n	q_n
0	1	1
1	0.9	0.9
2	0.81	0.729
3	0.729	0.4782969
4	0.6561	0.205891132094649
5	0.59049	0.0381520424476946
6	0.531441	0.00131002050863762
7	0.4782969	0.00000154453835975
8	0.43046721	0.0000000000021470

Fixed Point Convergence

- **Theorem 2.8**

  Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose g' is continuous on (a,b) and that $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a,b)$.

  If $g'(p) \neq 0$, then for all number p_0 in $[a,b]$, the sequence $p_n = g(p_{n-1})$ converges only **linearly** to the **unique fixed point** p in $[a,b]$.

- Proof:

  $$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p), \xi_n \in (p_n, p)$$

  Since $\{p_n\}_{n=0}^{\infty}$ converges to p, $\{\xi_n\}_{n=0}^{\infty}$ converges to p.

  Since g' is continuous, $\lim_{n\to\infty} g'(\xi_n) = g'(p)$

  $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|} = \lim_{n\to\infty} \left|g'\left(\xi_n\right)\right| = |g'(p)| \Rightarrow$ linear convergence

Speed up Convergence of Fixed Point Iteration

- If we look for faster convergence methods, we must have $g'(p) = 0$
- **Theorem 2.9**

  Let p be a solution of $x = g(x)$. Suppose $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that for $p_0 \in [p - \delta, p + \delta]$, the sequence defined by $p_{n+1} = g(p_n)$, when $n \geq 0$, converges ***at least quadratically*** to p. For sufficiently large n

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2$$

Remark:

Look for quadratically convergent fixed point methods which $g(p) = p$ and $g'(p) = 0.$

Newton's Method as Fixed-Point Problem

Solve $f(x) = 0$ by fixed point method. We write the problem as an equivalent fixed point problem:

$$g(x) = x - \phi(x)f(x) \quad \text{solve } x = g(x), \phi(x) \text{ is differentiable}$$

<u>Newton's method can be derived by the above form:</u>

Find differentiable $\phi(x)$ with $g'(p) = 0$ when $f(p) = 0$.

$$g'(x) = \frac{d}{dx}[x - \phi(x)f(x)] = 1 - \phi' f - \phi f'$$

Use $g'(p) = 0$ when $f(p) = 0$

$$g'(p) = 1 - \phi'(p) \cdot 0 - \phi(p)f'(p) = 0$$

$$\phi(p) = 1/f'(p)$$

This gives Newton's method

$$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)}$$

Example Fixed-point method and Newton's method are used to solve $\cos(x) - x = 0$ for $x \in [0,1]$, respectively. Compare the order of convergence of these two methods.

Multiple Roots

- How to modify Newton's method when $f'(p) = 0$. Here p is the root of $f(x) = 0$.
- **Definition 2.10 Multiplicity of a Root**

  A solution p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$, we can write $f(x) = (x - p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.
- **Theorem 2.11**

  $f \in C^1[a, b]$ has a **simple zero** at p in (a, b) if and only if $f(p) = 0$, but $f'(p) \neq 0$.
- **Theorem 2.12**

  The function $f \in C^m[a, b]$ has a zero of multiplicity m at point p in (a, b) if and only if

  $0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p)$, but $f^{(m)}(p) \neq 0$

Modified Newton's Method for Zeroes of Higher Multiplicity ($m > 1$)

Define the new function $\mu(x) = \frac{f(x)}{f'(x)}$.

Write $f(x) = (x-p)^m q(x)$, hence

$$\mu(x) = \frac{f(x)}{f'(x)} = (x-p)\frac{q(x)}{mq(x) + (x-p)q'(x)}$$

Note that p is a simple zero of $\mu(x)$.

- Apply Newton's method to $\mu(x) = 0$ to give:

$$x = g(x) \equiv x - \frac{\mu(x)}{\mu'(x)}$$

$$= x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

- Quadratic convergence: $p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{[f'(p_n)]^2 - f(p_n)f''(p_n)}$

Drawbacks:

- Compute $f''(x)$ is expensive
- Iteration formula is more complicated – more expensive to compute
- Roundoff errors in denominator – both $f'(x)$ and $f(x)$ approach zero.