2.5 Accelerating Convergence

Aitken's Δ^2 Method

- **Assume** $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p.
- Further assume $\frac{|p_{n+1}-p|}{|p_n - p|} \approx \frac{|p_{n+2}-p|}{|p_{n+1}-p|}$ when n is large
- Solving for p yields:

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

A little algebraic manipulation gives:

$$p \approx p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

- **Define** $\widehat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

Remark: The new sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster.

Definition

Aitken's Δ^2 Method: Given a sequence $\{p_n\}_{n=0}^{\infty}$ which converges to limit p. The new sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ defined by $\widehat{p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$ converges more rapidly to p than does the sequence $\{p_n\}_{n=0}^{\infty}$.

Remark:

1. numerator $p_{n+1} - p_n$ is a forward difference
2. denominator $p_{n+2} - 2p_{n+1} + p_n$ is central difference.

Example. Consider the sequence $\{p_n\}_{n=0}^{\infty}$ generated by the fixed point iteration $p_{n+1} = \cos(p_n), p_0 = 0.$

iteration	p_n	$\widehat{p_n}$
0	0.000000000000000	0.685073357326045
1	1.000000000000000	0.728010361467617
2	0.540302305868140	0.733665164585231
3	0.857553215846393	0.736906294340474
4	0.654289790497779	0.738050421371664
5	0.793480358742566	0.738636096881655
6	0.701368773622757	0.738876582817136
7	0.763959682900654	0.738992243027034
8	0.722102425026708	0.739042511328159
9	0.750417761763761	0.739065949599941
10	0.731404042422510	0.739076383318956
11	0.744237354900557	0.739081177259563*
12	0.735604740436347	0.739083333909684*

Remark: $\widehat{p_{11}}$ needs p_{13}; $\widehat{p_{12}}$ needs p_{14}. p_{13} and p_{14} are not shown here.

Theorem 2.14 Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to the limit p and that

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1$$

The Aitken's Δ^2 sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\widehat{p_n} - p}{p_n - p} = 0$$

Steffensen's Method

- Steffensen's Method is a combination of fixed-point iteration and the Aitken's Δ^2 method:

Step 0. Suppose we have a fixed point iteration:

$$p_0, \qquad \mathrm{p}_1 = \mathrm{g}(\mathrm{p}_0), \qquad \mathrm{p}_2 = \mathrm{g}(\mathrm{p}_1)$$

Once we have we have p_0, p_1 and p_2, we can compute

$$p_0^{(1)} = p_0 - \frac{(p_1 - p_0)^2}{(p_2 - 2p_1 + p_0)}$$

Step 1. Then we "restart" the fixed point iteration with

$$p_1^{(1)} = g\left(p_0^{(1)}\right), \quad p_2^{(1)} = g\left(p_1^{(1)}\right)$$

and compute:

$$p_0^{(2)} = p_0^{(1)} - \frac{\left(p_1^{(1)} - p_0^{(1)}\right)^2}{\left(p_2^{(1)} - 2p_1^{(1)} + p_0^{(1)}\right)}.$$

Step 2. We "restart" the fixed point iteration with

$$p_1^{(2)} = g\left(p_0^{(2)}\right), \quad p_2^{(2)} = g\left(p_1^{(2)}\right)$$

and compute:

$$p_0^{(3)} = p_0^{(2)} - \frac{\left(p_1^{(2)} - p_0^{(2)}\right)^2}{\left(p_2^{(2)} - 2p_1^{(2)} + p_0^{(2)}\right)}.$$

Illustration. Compare Fixed point iteration, Newton's method and Steffensen's method for solving: $f(x) = x^3 + 4x^2 - 10 = 0.$

Solution: $$x^3 + 4x^2 = 10$$

$$x^2(x+4) = 10$$

$$x^2 = \frac{10}{x+4}$$

Fixed point iteration: $p_{n+1} = g(p_n) = \sqrt{\frac{10}{p_n+4}}$

i	p_n	$g(p_n)$
0	1.50000	1.34840
1	1.34840	1.36738
2	1.36738	1.36496
3	1.36496	1.3652
4	1.36526	1.36523
5	1.36523	1.36523

2. Newton's method

i	x_n	$f(x_n)$
0	1.50000	1.51600e-01
1	1.36495	-3.11226e-04
2	1.36523	-1.35587e-09

3. Steffensen's method

$p_0^{(0)}$	$p_1^{(0)}$	$p_2^{(0)}$	$p_0^{(1)} = \{\Delta^2\}(p_0^{(0)})$	$\lvert p_2^{(0)} - p_0^{(1)}\rvert$
1.50000	1.34840	1.36738	1.36527	3.96903e-05
	$p_1^{(1)}$	$p_2^{(1)}$	$p_0^{(2)} = \{\Delta^2\}(p_0^{(1)})$	$\lvert p_2^{(1)} - p_0^{(2)}\rvert$
	1.36523	1.36523	1.36523	2.80531e-12