3.1 Interpolation and Lagrange Polynomial

Theorem 3.1 Weierstrass Approximation theorem

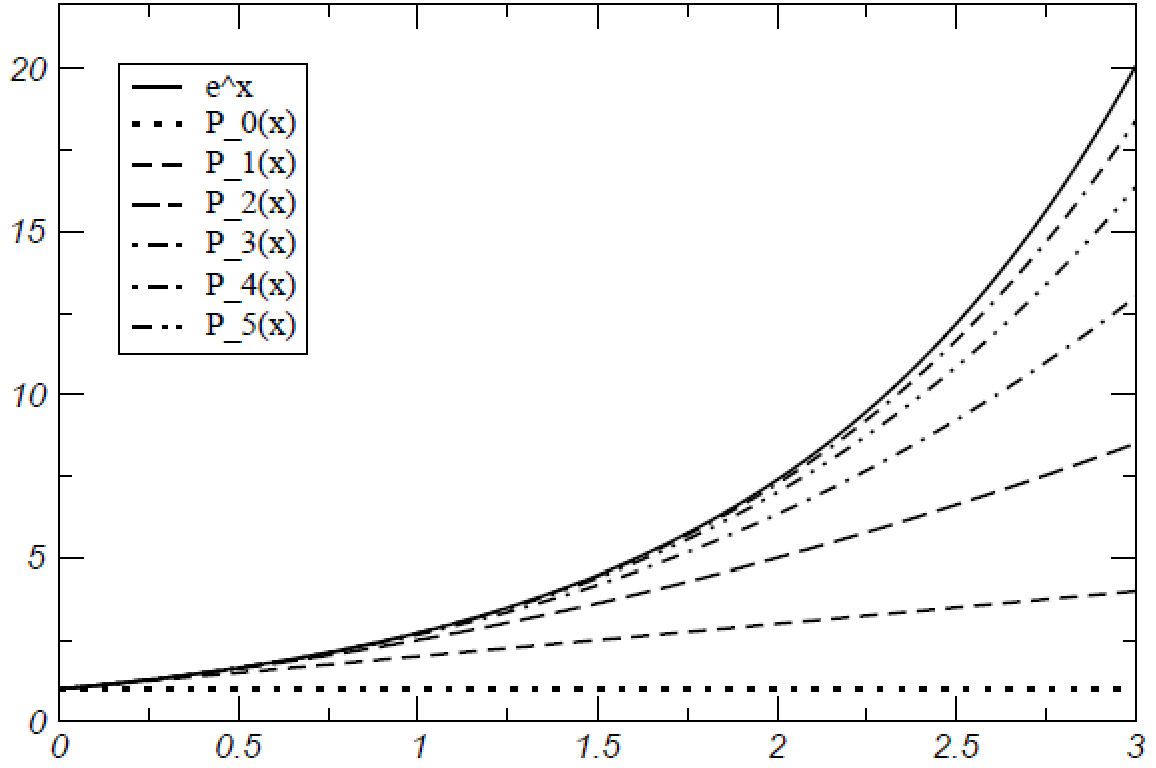
Suppose $f \in C[a,b]$. Then $\forall \epsilon > 0$, $\exists$ a polynomial $P(x)$:
$|f(x) - P(x)| < \epsilon, \forall x \in [a,b]$.

Remark:

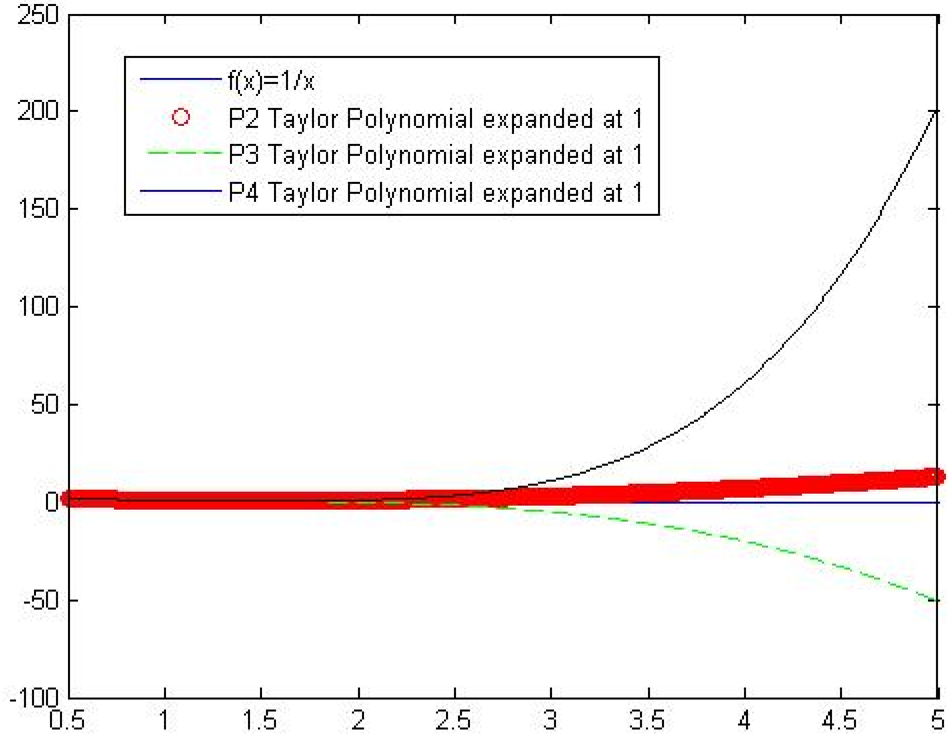
1. The bound is uniform, i.e. valid for all x in $[a,b]$
2. The way to find $P(x)$ is unknown.

- **Question:** Can Taylor polynomial be used here?
- Taylor expansion is accurate in the neighborhood of **one** point. We need to the (interpolating) polynomial to pass many points.

- **Example**. Taylor polynomial approximation of e^x for $x \in [0,3]$



- **Example**. Taylor polynomial approximation of $\frac{1}{x}$ for $x \in [0.5, 5]$. Taylor polynomials of different degrees are expanded at $x_0 = 1$



Exercise 3.1.2(a) Use nodes $x_0 = 1, x_1 = 1.25, x_2 = 1.6$ to find 2nd Lagrange interpolating polynomial $P(x)$for $f(x) = \sin(\pi x)$. And use $P(x)$ to approximate $f(1.4)$.

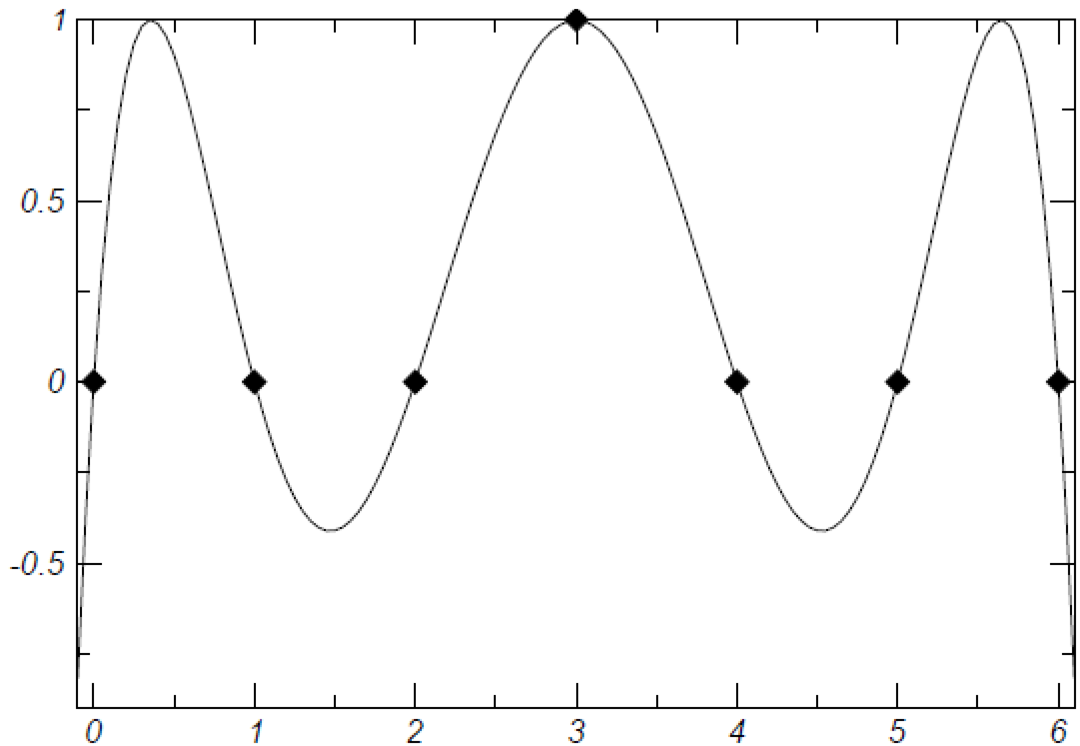
n-degree Interpolating Polynomial through $n+1$ Points

Constructing a Lagrange interpolating polynomial passing through the points $(x_0, f(x_0))$, $\big(x_1, f(x_1)\big)$, $\big(x_2, f(x_2)\big)$, … , $(x_n, f(x_n))$.

1. Define Lagrange basis functions $L_{n,k}(x) = \prod_{i=0,i\neq k}^{n} \frac{x-x_i}{x_k-x_i} = \frac{x-x_0}{x_k-x_0} \dots \frac{x-x_{k-1}}{x_k-x_{k-1}} \cdot \frac{x-x_{k+1}}{x_k-x_{k+1}} \dots \frac{x-x_n}{x_k-x_n}$ for $k = 0,1 \dots n$.

   Remark: $L_{n,k}(x_k) = 1; L_{n,k}(x_i) = 0,\ \forall i \neq k$

2. $P_n(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x)$.



- $L_{6,3}(x)$ for points $x_i = i,\ i = 0, \ldots, 6$.

- **Theorem 3.2** If $x_0, \dots, x_n$ are $n+1$distinct numbers (called nodes) and f is a function whose values are given at these numbers, then a **unique polynomial** $P(x)$ of **degree at most $\boldsymbol{n}$** exists with $P(x_k) = f(x_k)$, for each $k = 0, 1, \dots n$.

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x).$$

Where $L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$.

Exercise 3.1.6.a Construct 3rd degree interpolating polynomial with following tabulated values.

$x_0 = 0$	$\boldsymbol{f(0) = 1}$
$x_1 = 0.25$	$\boldsymbol{f(0.25) = 1.}64$
$x_2 = 0.5$	$\boldsymbol{f(0.5) = 2.71}$
$x_3 = 0.75$	$\boldsymbol{f(0.75) = 4.48}$

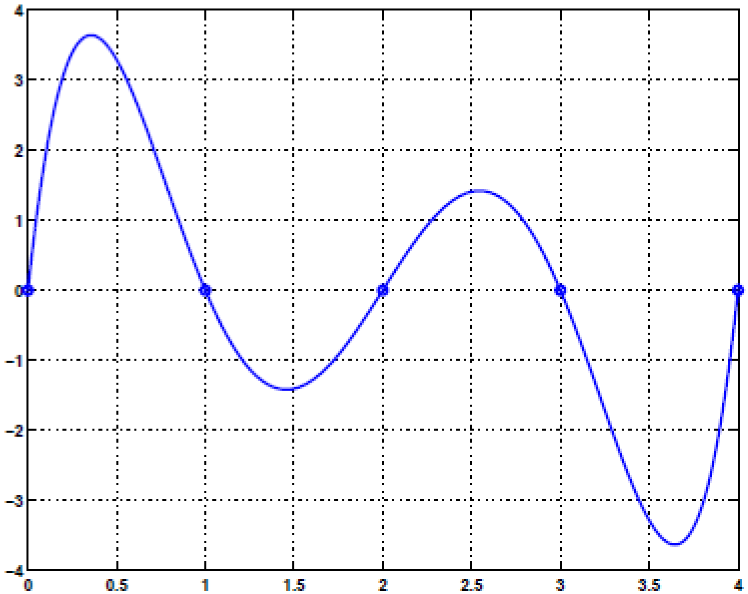
Error Bound for the Lagrange Interpolating Polynomial

Theorem 3.3 Suppose $x_0, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, \ldots, x_n$, and hence in (a, b), exists with $f(x) = P(x) +$

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n).$$

Where $P(x)$ is the Lagrange interpolating polynomial.

- Remark:
  1. Applying the error term may be difficult.
     $(x - x_0)(x - x_1) \ldots (x - x_n)$ is oscillatory.
     $\xi(x)$ is generally unknown.
  2. The error formula is important as they are used for numerical differentiation and integration.



Plot of $(x-0)(x-1)(x-2)(x-3)(x-4)$

Example 3 2nd Lagrange polynomial for $f(x) = \frac{1}{x}$ on [2, 4] using nodes $x_0 = 2, x_1 = 2.75, x_2 = 4$ is $P(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$. Determine the error form for $P(x)$, and maximum error when polynomial is used to approximate $f(x)$ for $x \in [2,4]$.

Example 4 Suppose a table is to be prepared for $f(x) = e^x, x \in [0,1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size is h. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0,1]$.