3.4 Hermite Interpolation
3.5 Cubic Spline Interpolation

Hermite Polynomial

Definition. Suppose $f \in C^1[a,b]$. Let $x_0, \dots, x_n$ be distinct numbers in $[a,b]$, the Hermite polynomial $P(x)$ approximating f is that:

1. $P(x_i) = f(x_i)$, for $i = 0, \dots, n$
2. $\frac{dP(x_i)}{dx} = \frac{df(x_i)}{dx}$, for $i = 0, \dots, n$

Remark: $P(x)$ and $f(x)$ agree not only function values but also 1st derivative values at x_i, $i = 0, \dots, n$.

Osculating Polynomials

Definition 3.8 Let $x_0, \dots, x_n$ be distinct numbers in $[a, b]$ and for $i = 0, \dots, n$, let m_i be a nonnegative integer. Suppose that $f \in C^m[a, b]$, where $m = \max\limits_{0 \le i \le n} m_i$. The osculating polynomial approximating f is the polynomial $P(x)$ of least degree such that $\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$ for each $i = 0, \dots, n$ and k$= 0, \dots, m_i$.

Remark: the degree of $P(x)$ is at most $M = \sum_{i=0}^{n} m_i + n$.

Theorem 3.9 If $f \in C^1[a,b]$ and $x_0, \dots, x_n \in [a,b]$ distinct numbers, the Hermite polynomial of degree at most $2n+1$ is:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

Where

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}(x_j)] L_{n,j}^2(x)$$
$$\hat{H}_{n,j}(x) = (x - x_j) L_{n,j}^2(x)$$

Moreover, if $f \in C^{2n+2}[a,b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \dots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

for some $\xi(x) \in (a,b)$.

Remark:

1. $H_{2n+1}(x)$ is a polynomial of degree at most $2n+1$.
2. $L_{n,j}(x)$ is jth Lagrange basis polynomial of degree n.
3. $\frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$ is the error term.

Example 3.4.1 Use Hermite polynomial that agrees with the data in the table to find an approximation of $f(1.5)$

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	-0.5220232
1	1.6	0.4554022	-0.5698959
2	1.9	0.2818186	-0.5811571

3rd Degree Hermite Polynomial

- Given distinct x_0, x_1 and values of f and f' at these numbers.

$$
\begin{aligned}
&H_3(x) \\
&= \left(1 + 2\,\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f(x_0) \\
&+ (x - x_0)\left(\frac{x_1 - x}{x_1 - x_0}\right)^2 f'(x_0) \\
&+ \left(1 + 2\,\frac{x_1 - x}{x_1 - x_0}\right)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f(x_1) \\
&+ (x - x_1)\left(\frac{x_0 - x}{x_0 - x_1}\right)^2 f'(x_1)
\end{aligned}
$$

Hermite Polynomial by Divided Differences

Suppose $x_0, \dots, x_n$ and f, f' are given at these numbers.
Define $z_0, \dots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \qquad \text{for } i = 0, \dots, n$$

Construct divided difference table, but use

$$f'(x_0), f'(x_1), .., f'(x_n)$$

to set the following undefined divided difference:

$$f[z_0, z_1], f[z_2, z_3], \dots, f[z_{2n}, z_{2n+1}].$$

The Hermite polynomial is

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0) \dots (x - z_{k-1})$$

Example 3.4.2 Use divided difference method to determine the Hermite polynomial that agrees with the data in the table to find an approximation of $f(1.5)$

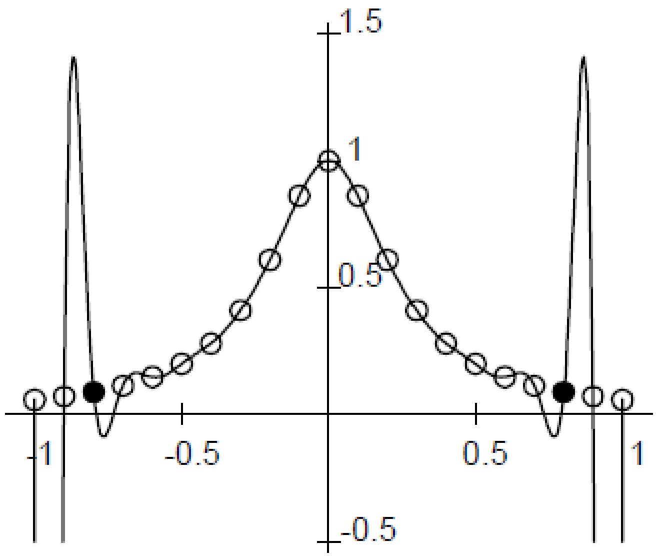
k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	−0.5220232
1	1.6	0.4554022	−0.5698959
2	1.9	0.2818186	−0.5811571

Divided Difference Notation for Hermite Interpolation

- Divided difference notation for Hermite polynomial interpolating 2 nodes: x_0, x_1.

$$H_3(x) = f(x_0) + f'(x_0)(x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 + f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1)$$

Problems with High Order Polynomial Interpolation



- 21 equal-spaced numbers to interpolate $f(x) = \frac{1}{1+25x^2}$. The interpolating polynomial oscillates between interpolation points.

3.5 Cubic Splines

- Idea: Use piecewise polynomial interpolation, i.e, divide the interval into smaller sub-intervals, and construct different low degree polynomial approximations (with small oscillations) on the sub-intervals.
- Challenge: If $f'(x_i)$ are not known, can we still generate interpolating polynomial with continuous derivatives?

Definition 3.10 Given a function f on $[a, b]$ and nodes $a = x_0 < \cdots < x_n = b$, a **cubic spline interpolant** S for f satisfies:

(a) $S(x)$ is a cubic polynomial $S_j(x)$ on $[x_j, x_{j+1}]$ with:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$
$$\forall j = 0,1, \dots, n - 1.$$

(a) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$, $\forall j = 0,1, \dots, n - 1.$

(b) $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$, $\forall j = 0,1, \dots, n - 2.$

Remark: (b) is derived from (a).

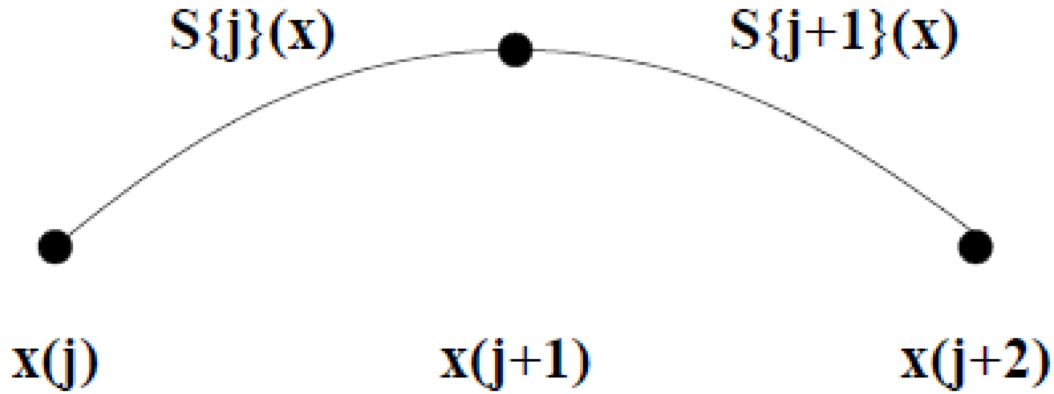
(c) $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$, $\forall j = 0,1, \dots, n - 2.$

(d) $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$, $\forall j = 0,1, \dots, n - 2.$

(e) One of the following boundary conditions:

(i) $S''(x_0) = S''(x_n) = 0$ (called free or natural boundary)

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (called clamped boundary)



Things to match at interior point x_{j+1}:

- The spline segment $S_j(x)$ is on $[x_j, x_{j+1}]$.
- The spline segment $S_{j+1}(x)$ is on $[x_{j+1}, x_{j+2}]$.
- Their function values: $S_j(x_{j+1}) = S_{j+1}(x_{j+1}) = f(x_{j+1})$
- First derivative values: $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$
- Second derivative values: $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$

Example 3.5.1 Construct a natural spline $S(x)$ through (1,2), (2,3) and (3.5).

Building Cubic Splines

- Define: $S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$
  and $h_j = x_{j+1} - x_j, \forall j = 0,1, \dots, n-1.$

Solve for coefficients a_j, b_j, c_j, d_j by:

1. $S_j(x_j) = a_j = f(x_j)$ for $j = 0,1,..,n-1$. We also define $a_n = f(x_n)$.
2. $S_{j+1}(x_{j+1}) = a_{j+1} = a_j + b_j h_j + c_j(h_j \;)^2 + d_j(h_j \;)^3$
   for $j = 0,1, \dots, n-1.$
   $Note$: $a_n = a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}(h_{n-1})^2 + d_{n-1}(h_{n-1})^3$
3. $S'_j(x_j) = b_j$, also $b_{j+1} = b_j + 2c_j h_j + 3d_j(h_j \;)^2$
   for $j = 0,1, \dots, n-1$ with b_n defined to be $b_n = S'(x_n)$
4. $S''_j(x_j) = 2c_j$, also $c_{j+1} = c_j + 3d_j h_j$
   for $j = 0,1, \dots, n-1$ with c_n defined to be $c_n = S''(x_n)/2$
5. Natural or clamped boundary conditions

Solving the Resulting Equations

$\forall j = 1,2,\ldots,n-1$

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) \qquad (3.21)$$

Remark: (n-1) equations for (n+1) unknowns $\{c_j\}_{j=0}^{n}$. Eq. (3.21) is solved with boundary condition.

- Once compute c_j, we then compute:

$$b_j = \frac{(a_{j+1} - a_j)}{h_j} - \frac{h_j(2c_j + c_{j+1})}{3} \qquad (3.20)$$

and

$$d_j = \frac{(c_{j+1} - c_j)}{3h_j} \qquad (3.17)$$ for $j = 0,1,2,\ldots,n-1$

Building Natural Cubic Spline

- Natural boundary condition:
  1. $0 = S''_0(x_0) = 2c_0 \rightarrow c_0 = 0$
  2. $0 = S''_n(x_n) = 2c_n \rightarrow c_n = 0$

1. Solve Eq. (3.21) together with (1) and (2).
2. Solve Eq. (3.20)
3. Solve Eq. (3.17)

Building Clamped Cubic Spline

- Clamped boundary condition:

  a) $S'_0(x_0) = b_0 = f'(x_0)$

  b) $S'_{n-1}(x_n) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n) = f'(x_n)$

  Remark: a) and b) gives additional equations:

  $$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(x_0) \qquad (a)$$

  $$h_{n-1}c_{n-1} + 2h_{n-1}c_n = -\frac{3}{h_0}(a_n - a_{n-1}) + 3f'(x_n) \qquad (b)$$

1. Solve Eq. (3.21) together with (a) and (b).
2. Solve Eq. (3.20)
3. Solve Eq. (3.17)

Example 3.5.4 Let $(x_0, f(x_0)) = (0,1), (x_1, f(x_1)) = (1, e), (x_2, f(x_2)) = (2, e^2)$, $(x_3, f(x_3)) = (3, e^3)$. And $f'(x_0) = e$, $f'^{(x_3)} = e^3$. Determine the clamped spline $S(x)$.

Theorem 3.11 If f is defined at the nodes: $a = x_0 < \cdots < x_n = b$, then f has a unique natural spline interpolant S on the nodes; that is a spline interpolant that satisfied the natural boundary conditions $S''(a) = 0, S''(b) = 0$.

Theorem 3.12 If f is defined at the nodes: $a = x_0 < \cdots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes; that is a spline interpolant that satisfied the clamped boundary conditions $S'(a) = f'(a), S'(b) = f'(b)$.

Error Bound

Theorem 3.13 If $f \in C^4[a, b]$, let $M = \max_{a \leq x \leq b} |f^4(x)|$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes: $a = x_0 < \cdots < x_n = b$, then with

$$h = \max_{0 \leq j \leq n-1} \left(x_{j+1} - x_j\right)$$

$$\max_{a \leq x \leq b} |f(x) - S(x)| \leq \frac{5Mh^4}{384}.$$